Math 128b, problem set 05
Due: Fri Mar 03

Problems to be done, but not turned in: (Ch. 17) 29, 31, 33; (Ch. 18) 1, 3, 5, 7, 13, 15, 17, 19, 23, 29.

Fun: 32.

Problems to be turned in:

1. (Ch. 18) 4.
2. (Ch. 18) 12.
3. (Ch. 18) 18.
4. (Ch. 18) 22.
5. Factor 2 into irreducibles in the rings $\mathbf{Z}$, $\mathbf{Z}[\sqrt{2}]$, $\mathbf{Z}[\sqrt{2}, \sqrt[4]{2}]$, $\mathbf{Z}[\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}]$, etc. Give an example of an integral domain D in which there exists some $a \in D$ that *cannot* be factored into irreducible elements of D.
6. Let D be an integral domain such that every $a \in D$ can be factored into irreducible elements of D. Prove that D is a unique factorization domain if and only if every irreducible element of D is prime.
7. Let D be a unique factorization domain.
   (a) Let $a = p_1 \dots p_r$ and $b = q_1 \dots q_s$ be factorizations of $a, b \in D$ into irreducibles p_i and q_i. Prove that a divides b if and only if $r \leq s$ and, after possibly reordering the p_i and q_j, p_i is an associate of q_i for $1 \leq i \leq r$.
   (b) Let a and b be nonzero elements of D. Prove that there exists $d \in D$ such that d divides a, d divides b, and if $e \in D$ divides both a and b, then e divides d. We call d the *greatest common divisor* of a and b, and denote it by $\gcd(a, b)$.
   (c) Now assume that D is a principal idea domain. Describe $d = \gcd(a, b)$ in terms of the ideals $\langle a \rangle$ and $\langle b \rangle$.