Supplemental notes on chapter 6
Math 129b

The Whatever Theorem. This says:

The Whatever Theorem. *Let V and W be vector spaces, let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V, and let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be vectors in W (possibly equal to each other or $\mathbf{0}$). Then there exists a unique linear function $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for $1 \leq i \leq n$.*

The main ideas of the Whatever Theorem are: (1) You can make a linear function do Whatever you want to a basis, and (2) This is essentially the only way to make up a linear function/write down a formula for a linear function.

The SPAM and One-to-one Lemmas. These are somewhat complementary tools for proving facts about a linear function T. The SPAM Lemma can be used to prove T is onto, or other facts about the image of T, by finding a SPAnning set for the iMage of T. The One-to-one Lemma deals with the kernel of T.

The SPAM Lemma. *If $T : V \to W$ is linear and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ spans $\operatorname{im} T$.*

The One-to-one Lemma. *If $T : V \to W$ is linear, then the following are equivalent:*
1. *T is one-to-one.*
2. *$\ker T = \{\mathbf{0}\}$.*
3. *$\operatorname{nullity} T = 0$.*

The matrix of a linear function. The matrix of the linear function T relative to the bases B (domain) and B' (range) is denoted by $[T]_{B,B'}$. Let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Then by definition, we have

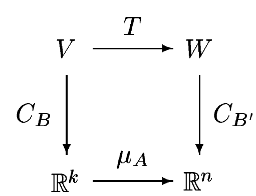
$$[T]_{B,B'} = \Big[[T(\mathbf{u}_1)]_{B'} \ \cdots \ [T(\mathbf{u}_k)]_{B'} \Big].$$

Slogan: "**The columns tell you where your basis goes.**"

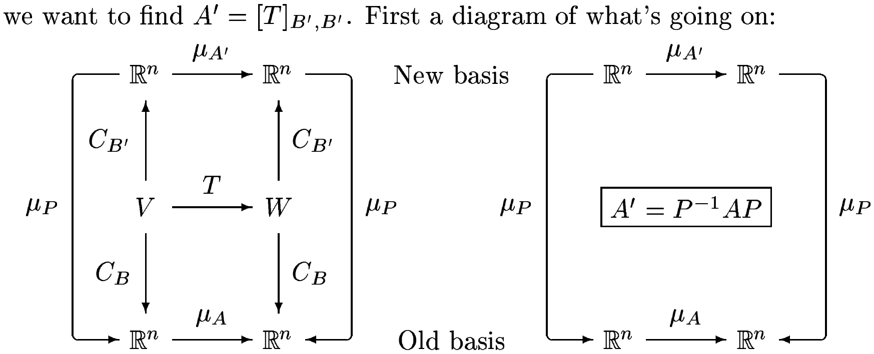
The key property of $A = [T]_{B,B'}$ is that

$$A(B\text{-coordinates of } \mathbf{v}) = B'\text{-coordinates of } T(\mathbf{v}).$$

Diagram:



Change of basis. Suppose $T : V \to V$ is linear, and that we know the matrix $A = [T]_{B,B}$ of T relative to an old basis B. Suppose we want to find the matrix of T relative to some new basis B', i.e., suppose we want to find $A' = [T]_{B',B'}$. First a diagram of what's going on:



Definition. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the old basis, and let $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ be the new basis. The *change-of-basis matrix from the basis B' to the basis B* is the matrix P whose ith column is $[\mathbf{v}'_i]_B$:

$$P = \left[[\mathbf{v}'_1]_B \ \cdots \ [\mathbf{v}'_n]_B \right].$$

The key property of P is that

$$P(B'\text{-coordinates of } \mathbf{v}) = B\text{-coordinates of } \mathbf{v}.$$

Then the formula in the change-of-basis theorem is:

$$[T]_{B',B'} = A' = P^{-1}AP.$$

The most important case: Suppose $V = \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, and the old basis is the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Mercifully, in that case, $[T]_{S,S} = A$.

If we want to find the matrix of T relative to a new basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then P changes B-coordinates to S-coordinates (mnemonic: "change of basis is just a bunch of B.S."), and the columns of P are just the vectors in B.