**Supplemental notes on chapter 8**
**Math 129b**

Combining 8.12, 8.13, 8.15, and some other results and definitions, we have:

**The Diagonalization Theorem.** *Let $V$ be a vector space of dimension $n$, and let $T : V \to V$ be linear. Then the following are equivalent:*

1. *There exists a basis $B = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ for $V$ such that each $\mathbf{u}_i$ is an eigenvector of $T$.*
2. *There exists a basis $B$ for $V$ such that $[T]_{B,B}$ is diagonal.*
3. *The characteristic polynomial of $T$ factors as*

$$\text{char poly}(T) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_r)^{m_r}, \tag{1}$$

*where $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of $T$; and furthermore, for each distinct eigenvalue $\lambda_i$,*

$$\dim E_T(\lambda_i) = m_i = \text{multiplicity}(\lambda_i). \tag{2}$$

4. *We have that*

$$\sum \dim E_T(\lambda_i) = n, \tag{3}$$

*where the sum runs over all distinct eigenvalues $\lambda_i$ of $T$.*

*Moreover, when these conditions are true, for the basis $B$ in conditions 1 and 2, we have that*

$$[T]_{B,B} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{4}$$

*where $\lambda_i$ is the eigenvalue of $\mathbf{u}_i$ (and here, some of the $\lambda_i$ may be repeated).*

**Remark.** Note that condition (2) holds automatically for every eigenvalue $\lambda_i$ of multiplicity 1, since in that case,

$$1 \le \dim E_T(\lambda_i) \le \text{multiplicity}(\lambda_i) = 1, \tag{5}$$

where the first $\le$ holds because eigenvectors are nonzero, and the second $\le$ is a result from the problem sets.

*Proof.* **(1⇒2).** Assume condition 1. This boils down to the computation of $[T]_{B,B}$; see the proof of Thm. 8.12 on p. 315.

**(2⇒3).** Assume condition 2. If necessary, reorder the vectors in $B$ so that repeated diagonal entries of $[T]_{B,B}$ are grouped together. We may therefore assume that

$$A = [T]_{B,B} = \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \lambda_r & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_r \end{bmatrix}, \tag{6}$$

where $r$ is the number of distinct diagonal entries that appear. Let $m_i$ be the number of times that $\lambda_i$ appears in the diagonal of $A$. An easy calculation (exercise) then shows that the characteristic polynomial of $A$, and therefore, the characteristic polynomial of $T$, has the form shown in (1). It follows that the (distinct) eigenvalues of $T$ are precisely $\lambda_1, \dots, \lambda_r$, and the multiplicity of $\lambda_i$ is precisely $m_i$.

It therefore remains only to verify (2). However, $\dim E_T(\lambda_i) = \dim E_A(\lambda_i) = \text{nullity}(A - \lambda_i I_n)$ and

$$A - \lambda_i I_n = \begin{bmatrix} \lambda_1 - \lambda_i & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 - \lambda_i & & & & & & \\ & & & \ddots & & & & & \\ & & & & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \lambda_r - \lambda_i & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \lambda_r - \lambda_i \end{bmatrix}, \tag{7}$$

where the 0's appear precisely where the $\lambda_i$'s appear in $A$, and the other diagonal entries $\lambda_j - \lambda_i$ are nonzero because $\lambda_j \neq \lambda_i$. It follows that $\dim E_T(\lambda_i) = m_i$.

**(3⇒4).** Assume condition 3. Then by (2), we see that $\sum_{i=1}^{r} \dim E_T(\lambda_i) = \sum_{i=1}^{r} m_i$, and by comparing degrees of polynomials in (1), we see that $\sum_{i=1}^{r} m_i = n$.

**(4⇒1).** Assume condition 4. Let $d_i = \dim E_T(\lambda_i)$, and let $\{\mathbf{u}_{i1}, \dots, \mathbf{u}_{id_i}\}$ be a basis for $E_T(\lambda_i)$. By TIDE (The Independence of Distinct Eigenstuff, Thm. 8.14), we see that

$$B = \{\mathbf{u}_{11}, \dots, \mathbf{u}_{1d_1}, \dots, \mathbf{u}_{r1}, \dots, \mathbf{u}_{rd_r}\} \tag{8}$$

is linearly independent. However, since the number of vectors in $B$ is then $\sum \dim E_T(\lambda_i) = n$, by the Two out of Three Theorem, $B$ is actually a basis for $V$. □

**Corollary.** *Let $V$ be a vector space of dimension $n$, and let $T : V \to V$ be linear. We do* **not** *assume that $T$ is diagonalizable; in particular, we do not assume that the equivalent conditions in the Diagonalization Theorem are true. Nevertheless, we still have that*

$$\sum \dim E_T(\lambda_i) \leq n, \tag{9}$$

*where the sum runs over all distinct eigenvalues $\lambda_i$ of $T$.*

*Proof.* Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of $T$. Even without assuming condition 4 of the Diagonalization Theorem, all of the proof of (4⇒1) still works except the last sentence. In particular, the set

$$B = \{\mathbf{u}_{11}, \dots, \mathbf{u}_{1d_1}, \dots, \mathbf{u}_{r1}, \dots, \mathbf{u}_{rd_r}\} \tag{10}$$

is still linearly independent and contains $\sum \dim E_T(\lambda_i)$ vectors. Therefore, by the comparison theorem, $\sum \dim E_T(\lambda_i)$, which is the number of vectors in the linearly independent set $B$, is at most $\dim V = n$. □

By the Change-of-Basis Theorem, we may translate the linear function version of the Diagonalization Theorem into:

**The Diagonalization Theorem (Matrix version).** *Let $A$ be an $n \times n$ matrix. Then the following are equivalent:*

1. *There exists a basis $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$ such that each $\mathbf{u}_i$ is an eigenvector of $A$.*
2. *There exists an invertible $n \times n$ matrix $P$ such that $P^{-1}AP$ is diagonal.*
3. *The characteristic polynomial of $A$ factors as*
$$\text{char poly}(A) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_r)^{m_r}, \tag{11}$$
*where $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of $A$; and furthermore, for each distinct eigenvalue $\lambda_i$,*
$$\dim E_A(\lambda_i) = m_i = \text{multiplicity}(\lambda_i). \tag{12}$$
4. *We have that*
$$\sum \dim E_A(\lambda_i) = n, \tag{13}$$
*where the sum runs over all distinct eigenvalues $\lambda_i$ of $A$.*

*Moreover, when these conditions are true, for the basis $B$ in condition 1 and the matrix $P$ in condition 2, we may take the columns of $P$ to be the vectors in $B$, in which case*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \tag{14}$$

*where $\lambda_i$ is the eigenvalue of $\mathbf{u}_i$ (and here, some of the $\lambda_i$ may be repeated).*