Math 129b, problem set 02
Outline due: Wed Sep 08
Due: Mon Sep 13
Last revision due: Fri Oct 15

Important terms and symbols:

restriction For X, Y sets, $f : X \to Y$, and $X' \subseteq X$, the *restriction* $f|_{X'}$ of f to X' is the function $f : X' \to Y$ defined by $f|_{X'}(x) = f(x)$ for all $x \in X'$. (I.e., same formula, smaller domain.)

Problems to be done, but not turned in: (1.7) 5, 8, 11; (1.8) 3, 5, 7, 11, 16, 19.

Problems to be turned in:

1. Let X be a set. Prove that $\mathbb{F}(X)$ satisfies axioms 3 and 4 of a vector space, being careful about the notation. (Suggestions: For axiom 3, you need to carefully define an appropriate function $\mathbf{0}$. Similarly, for axiom 4, you need to define a function $-f$ at an appropriate time.)

2. In both parts of this question, assume that $f, g \in C(\mathbb{R})$ (the set of all continuous real-valued functions on $\mathbb{R}$).

   (a) Is it possible to find $f, g \in C(\mathbb{R})$ such that $f(x) = g(x)$ for all $x < 0$ but $f \neq g$? Give an example or explain why not.

   (b) Is it possible to find $f, g \in C(\mathbb{R})$ such that $f(7) \neq g(7)$ but $f = g$? Give an example or explain why not.

3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Let W be the set of all 3×2 matrices X such that $AX = 0$ (the 3×2 zero matrix); in other words, let
$$W = \{X \in \mathbb{M}(3,2) \mid AX = 0\}.$$
   Prove that W is a subspace of $\mathbb{M}(3,2)$.

4. Let W be the set of all continuous functions on the interval $[0,1]$ with integral 0; in other words, let
$$W = \left\{ f \in \mathbb{C}([0,1]) \mid \int_0^1 f(x)\,dx = 0 \right\}.$$
   Is W a subspace of $\mathbb{C}([0,1])$? Prove or disprove.

5. Let W be the set of all 2×2 matrices with determinant 0; in other words, let
$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(2,2) \mid ad - bc = 0 \right\}.$$
   Is W a subspace of $\mathbb{M}(2,2)$? Prove or disprove.

(cont. on other side)

6. A *plane* in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ of the form

$$S(a,b,c,d) = \left\{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d\right\},$$

where $a, b, c, d \in \mathbb{R}$, and at least one of a, b, c is $\neq 0$.

(a) Find $a, b, c, d \in \mathbb{R}$ such that $S(a,b,c,d)$ is a subspace of $\mathbb{R}^3$, and find $a, b, c, d \in \mathbb{R}$ such that $S(a,b,c,d)$ is not a subspace of $\mathbb{R}^3$. (No proof necessary.)

(b) Find and prove the best possible theorem of the form "For a fixed a, b, c, d, the plane $S(a,b,c,d)$ defined above is a subspace of $\mathbb{R}^3$ if and only if (insert appropriate condition here)."