Math 129b, problem set 06
Outline due: Wed Oct 20
Due: Mon Oct 25
Last revision due: Fri Nov 12

Problems to be done, but not turned in: (6.2) 1, 5, 10, 15, 20; (6.6) 1, 5, 7.

Problems to be turned in:

1. (6.2) 12.
2. (6.2) 13.
3. Recall that $\mathbb{P}_4$ is the vector space of all polynomials of degree ≤ 4; in particular, an arbitrary element of $\mathbb{P}_4$ has the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ $(a_i \in \mathbb{R})$. Let $T : \mathbb{P}_4 \to \mathbb{P}_4$ be defined by the formula

$$T(p(x)) = p(x) - p(2)$$

for all $p(x) \in \mathbb{P}_4$.

   (a) Prove that T is linear.

   (b) Find a basis for $\ker T$, a basis for $\operatorname{im} T$, the rank of T, and the nullity of T.

4. Recall that $\mathbb{P}$ is the vector space of all polynomials (of any degree). Define linear maps $D : \mathbb{P} \to \mathbb{P}$ and $I : \mathbb{P} \to \mathbb{P}$ by the formulas

$$D(p(x)) = p'(x), \qquad I(p(x)) = \int_0^x p(t)\,dt.$$

In other words, D is differentiation, and I is indefinite integration, choosing the constant $C = 0$.

   (a) Give as precise a description as possible of exactly which polynomials are in $\ker D$ and $\operatorname{im} D$. (I.e., your description should let a reader know which polynomials are in $\ker D$ and $\operatorname{im} D$ without requiring the reader to do any computation.) Is D one-to-one? Is D onto?

   (b) Give as precise a description as possible of exactly which polynomials are in $\ker I$ and $\operatorname{im} I$. Is I one-to-one? Is I onto?

5. Let $L : U \to V$ and $T : V \to W$ be linear, and assume that U, V, and W are all different vector spaces. Label each of the following statements "always true," "sometimes true, sometimes false," or "always false." Justify/prove each answer.

   (a) $\ker T \subseteq \ker T \circ L$.

   (b) $\ker L \subseteq \ker T \circ L$.

   (c) $\operatorname{im} T \circ L \subseteq \operatorname{im} T$.

   (d) $\operatorname{im} T \circ L \subseteq \operatorname{im} L$.

(cont. on other side)

6. Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear. Prove that exactly one of the following is true:

- The equation $T(\mathbf{x}) = \mathbf{b}$ has a solution $\mathbf{x} \in V$ for all $\mathbf{b} \in V$.
- $\text{nullity}(T) > 0$.

(Aside: This theorem is known as the *Fredholm Alternative*.)