

Math 129b, problem set 07
Outline due: Wed Oct 27
Due: Mon Nov 01
Last revision due: Fri Nov 12

Important terms and symbols:

matrix of a linear function The matrix of the linear function T relative to the bases B (domain) and B' (range) is denoted by $[T]_{B,B'}$. By definition, this is the matrix $A = [T]_{B,B'}$ such that

$$A(B\text{-coordinates of } \mathbf{v}) = B'\text{-coordinates of } T(\mathbf{v}).$$

More precisely, if $B = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, then we have

$$[T]_{B,B'} = \begin{bmatrix} [T(\mathbf{u}_1)]_{B'} & \cdots & [T(\mathbf{u}_k)]_{B'} \end{bmatrix}.$$

That is, the i th column of $[T]_{B,B'}$ is $[T(\mathbf{u}_i)]_{B'}$.

Problems to be done, but not turned in: (6.7) 1, 9; (6.8) 2, 4; (6.3) 1, 11.

Problems to be turned in:

1. Let $T : V \rightarrow W$ be an isomorphism (of vector spaces), and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . Prove that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis for W .
2. (a) Does there exist a linear function $S : \mathbb{R}^3 \rightarrow \mathbb{P}_4$ such that S is one-to-one? If so, write down a formula for one such S , and prove that your S is one-to-one; if not, prove that no such S can exist.
(b) Let

$$V = \text{span} \{1, \cos x, \sin x, e^x, e^{2x}\} \subseteq \mathbb{F}(\mathbb{R}).$$

(You may assume that $\{1, \cos x, \sin x, e^x, e^{2x}\}$ is linearly independent.) Does there exist a linear function $T : V \rightarrow \mathbb{M}(3, 2)$ such that T is onto? If so, write down a formula for one such T , and prove that your T is onto; if not, prove that no such T can exist.

- (c) Does there exist a linear function $L : \mathbb{P}_4 \rightarrow \mathbb{M}(3, 2)$ such that $\text{rank } L = 3$? If so, write down a formula for one such L , and prove that $\text{rank } L = 3$; if not, prove that no such L can exist.
3. Let V and W be vector spaces, and let U be a subspace of W such that $\dim V = 5$, $\dim W = 7$, and $\dim U = 3$. Prove that there exists a linear function $T : V \rightarrow W$ such that $\text{nullity } T = 1$ and $\dim(U \cap (\text{im } T)) = 2$. (Note that as part of your proof, you must prove that $\text{nullity } T$ and $\dim(U \cap (\text{ker } T))$ are what you claim they are.)

(cont. on other side)

4. Let $D : \mathbb{P}_3 \rightarrow \mathbb{P}_2$ be defined by the formula

$$D(p(x)) = p'(x).$$

- (a) Recall that $B_1 = \{1, x, x^2, x^3\}$ is an ordered basis for \mathbb{P}_3 and $B_2 = \{1, x, x^2\}$ is an ordered basis for \mathbb{P}_2 . Find $[D]_{B_1, B_2}$, the matrix of D relative to B_1 and B_2 .
 - (b) It can be shown (i.e., take it as given) that $B_3 = \{1, x + 1, (x + 1)^2, (x + 1)^3\}$ is an ordered basis for \mathbb{P}_3 . Find $[D]_{B_3, B_2}$, the matrix of D relative to B_3 and B_2 .
 - (c) It can be shown (i.e., take it as given) that $B_4 = \{1, x + 1, (x + 1)^2\}$ is an ordered basis for \mathbb{P}_2 . Find $[D]_{B_1, B_4}$, the matrix of D relative to B_1 and B_4 .
5. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis for \mathbb{R}^k , and let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be k vectors in \mathbb{R}^n . Recall that the Whatever Theorem implies that there exists a unique linear $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that $T(\mathbf{u}_i) = \mathbf{w}_i$, and that Thm. 6.10 implies that $T = \mu_D$ for some $n \times k$ matrix D . The point of this problem is to find an explicit formula for D .
- (a) Describe the matrix A such that $A\mathbf{e}_i = \mathbf{u}_i$ for $1 \leq i \leq k$, with justification. What size is A ?
 - (b) Describe the matrix B such that $B\mathbf{u}_i = \mathbf{e}_i$ for $1 \leq i \leq k$, with justification. What size is B ?
 - (c) Describe the matrix C such that $C\mathbf{e}_i = \mathbf{w}_i$ for $1 \leq i \leq k$, where $\mathbf{e}_i \in \mathbb{R}^k$. What size is C ?
 - (d) Describe the matrix D such that $D\mathbf{u}_i = \mathbf{w}_i$ for $1 \leq i \leq k$, with justification. What size is D ?