Math 129b, problem set 08
Outline due: Wed Nov 03
Due: Mon Nov 08
Last revision due: Wed Dec 08

Problems to be done, but not turned in: (6.4) 2, 6; (6.5) 1, 5, 6; (8.1) 2, 9, 11, 15.

Problems to be turned in:

1. Let $T : V \to W$ be a linear function such that $\dim V = 6$, $\dim W = 4$, and $\operatorname{rank} T = 2$. Note that T is given; you do not get to choose or change T.

   Prove that there exist bases B for V and B' for W such that $[T]_{B,B'}$, the matrix of T relative to B and B', is

$$[T]_{B,B'} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (Note: Something similar can be proven for *any* linear function with finite-dimensional range and domain. In fact, we will see an even better version of this theorem when we study the *singular value decomposition*, near the end of the semester.)

2. For a fixed positive integer n, let $T_n : \mathbb{P}_n \to \mathbb{P}_n$ be defined by the formula

$$T_n(p(x)) = p(x-1),$$

   and let $B = \{1, \dots, x^n\}$ be the standard basis for $\mathbb{P}_n$.

   (a) Describe $[T_n]_{B,B}$, the matrix of T_n relative to the basis B. (Justify your answer.)

   (b) Find an explicit formula for T_n^{-1}, and use that formula to describe $[T_n^{-1}]_{B,B}$, the matrix of T_n^{-1} relative to the basis B. (Justify your answers.)

   (c) Interpret the previous parts of this problem purely in terms of matrix multiplication. (Justify your answer.)

(cont. on other side)

3. Let $A = \begin{bmatrix} 14 & -18 \\ 9 & -\frac{23}{2} \end{bmatrix}$, and let $T = \mu_A$ be the usual linear function associated with A. Note that $B = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. (This follows because neither vector is a scalar multiple of the other, making B linearly independent, and therefore a basis for $\mathbb{R}^2$, by the Two out of Three Theorem.)

   (a) Find $A' = [T]_{B,B}$, the matrix of T relative to the bases B and B.

   (b) Express $A' = [T]_{B,B}$ in terms of the original matrix A.

   (c) What happens to $(A')^n$ as $n \to \infty$?

   (d) Find a vector $\mathbf{v} \in \mathbb{R}^2$ such that, for any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, $A^n \begin{bmatrix} x \\ y \end{bmatrix}$ becomes arbitrarily close to a scalar multiple of $\mathbf{v}$ as $n \to \infty$.

4. Let $A = \begin{bmatrix} -8 & 20 & -10 \\ -5 & 12 & -5 \\ -5 & 10 & -3 \end{bmatrix}$. A calculation shows that the characteristic polynomial of A is $(\lambda - 2)^2(\lambda + 3)$ (you may assume this). Find the eigenvalues of A, and for each eigenvalue of A, find a basis for the associated eigenspace.

5. Let $V = \mathbb{D}^\infty(\mathbb{R})$, the space of infinitely differentiable functions on $\mathbb{R}$, and let $\Delta : V \to V$ be the linear function defined by $\Delta(f) = f''$. Prove that every $\lambda \in \mathbb{R}$ is an eigenvalue of Δ.

   Remark: In this context, Δ is often called the *Laplacian* on $\mathbb{R}$. The eigenvalues and eigenvectors (or in this context, *eigenfunctions*) of Δ play an important role in solving the equations governing heat, light, and sound, among other things.

6. Suppose that A is an $n \times n$ matrix such that $A^4 = I_n$. What are the possible eigenvalues of A? Prove your answer.