

Math 129b, problem set 09
Outline due: Wed Nov 17
Due: Mon Nov 22
Last revision due: Wed Dec 08

Important terms and symbols:

multiplicity Let A be an $n \times n$ matrix, and suppose that c is an eigenvalue of A . We define the *multiplicity* of c to be the largest integer m such that $(\lambda - c)^m$ divides the characteristic polynomial of A .

Problems to be done, but not turned in: (8.2) 1, 3, 9, 12, 15; (8.3) 2, 5, 7.

Problems to be turned in:

1. (8.2) 16, parts c and d only.
2. Fix a constant $c \in \mathbb{R}$.
 - (a) Let A be an $n \times n$ matrix, and suppose that $k = \dim E_A(c) > 0$. Prove that there exists an invertible matrix P such that columns $1, \dots, k$ of $P^{-1}AP$ are $c\mathbf{e}_1, \dots, c\mathbf{e}_k$, respectively.
 - (b) Let A be an $n \times n$ matrix, and suppose that c is an eigenvalue of A . Use part (a) to prove that $\dim E_A(c) \leq \text{multiplicity}(c)$.
3. Consider the matrices

$$A_1 = \begin{bmatrix} -5 & 0 & -6 \\ 3 & 1 & 3 \\ 3 & 0 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 & -3 \\ -2 & 3 & 6 \\ 4 & -1 & -5 \end{bmatrix}.$$

You may take it as given that the characteristic polynomials of A_1 and A_2 are both equal to $(x - 1)^2(x + 2)$. For each of the matrices A_i , either find a matrix P such that $P^{-1}A_iP$ is diagonal, or prove that no such P exists.

4. Suppose A is an $n \times n$ matrix such that $A(A - 3I_n) = 0$. (Note that $(A - 3I_n)A = A^2 - 3A = A(A - 3I_n) = 0$ as well.)
 - (a) Prove that every $\mathbf{v} \in \mathbb{R}^n$ is a linear combination of $A\mathbf{v}$ and $(A\mathbf{v} - 3\mathbf{v})$.
 - (b) Prove that if λ is an eigenvalue of A , then either $\lambda = 0$ or $\lambda = 3$.
 - (c) Prove that for $\mathbf{v} \in \mathbb{R}^n$, $(A\mathbf{v} - 3\mathbf{v}) \in E_A(0)$ and $A\mathbf{v} \in E_A(3)$. (Suggestion: Think of $E_A(0)$ and $E_A(3)$ as nullspaces.)
 - (d) Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $E_A(0)$, and let $\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ be a basis for $E_A(3)$. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ spans \mathbb{R}^n .
 - (e) Prove that A is diagonalizable, and describe what a diagonal matrix similar to A will look like.

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5. Let V be a vector space such that $\dim V = 4$, and let $T : V \rightarrow V$ and $L : V \rightarrow V$ be linear. Suppose there exist nonzero vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ such that $\mathbf{w}, \mathbf{x} \in E_T(3)$, $\mathbf{y}, \mathbf{z} \in E_T(-7)$, $\mathbf{w}, \mathbf{y} \in E_L(11)$, and $\mathbf{x}, \mathbf{z} \in E_L(-13)$. Prove that there exists a basis B for V such that $[T]_{B,B}$ and $[L]_{B,B}$ are diagonal. (In particular, describe $[T]_{B,B}$ and $[L]_{B,B}$.)