Sample exam 2
Math 129b, Fall 2004

You will not be allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything which has been proven in class, in the homework, or in the reading.

1. (12 points) Let A and B be sets, and let $f : A \to B$ be a function. Define what it means for f to be one-to-one, and define what it means for f to be onto. (Make sure the distinction is clear.)

2. (5 points) Let $B = \{1, \sin x, \cos x\} \subseteq \mathbb{F}(\mathbb{R})$, and let $W = \operatorname{span} B$. You make take it as given that B is an ordered basis for W. Let $f(x) = -7\cos x + 3$. Find $[f(x)]_B$, the coordinate vector of $f(x)$ relative to the ordered basis B. Show all your work.

For questions 3–6, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer as specifically as possible. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (12 points) Let U_1 and U_2 be subspaces of a vector space V, let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ be vectors in $U_1 \cap U_2$, and let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be vectors in U_2. It is possible that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ is linearly independent and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ spans U_2.

4. (12 points) It is possible to find polynomials $p_1(x), p_2(x), p_3(x) \in \mathbb{P}_2$ such that $\left\{1 + x + x^2, p_1(x), p_2(x), p_3(x)\right\}$ spans $\mathbb{P}_2$.

5. (12 points) Let A, B, C be nonzero elements of $\mathbb{M}(2,2)$. If A is not a linear combination of B and C, then $\{A, B, C\}$ must be linearly independent.

6. (12 points) Let V and W be vector spaces, let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be bases for V and W, respectively, and let $T : V \to W$ be linear. It is possible that $T(\mathbf{v}_1) = \mathbf{w}_1$, $T(\mathbf{v}_2) = \mathbf{w}_2$, $T(\mathbf{v}_3) = \mathbf{w}_3$, and $T(\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3) = \mathbf{0}$.

7. (15 points) **PROOF QUESTION.** Let V be a vector space, and let $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$ be vectors such that $\{\mathbf{v}, \mathbf{w}, \mathbf{x}\}$ spans V. Prove that $\{\mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{x}\}$ spans V.

8. (20 points) **PROOF QUESTION.** Let W_1 and W_2 be subspaces of $\mathbb{F}(\mathbb{R})$ such that $\dim W_1 = 6$, $\dim W_2 = 3$, and $W_2 \subseteq W_1$. Also, let f, g be elements of W_2 such that

$$f(1) = 3, \qquad f(2) = 7, \qquad g(1) = 0, \qquad g(2) = -1.$$

Prove that there exist $h_1, h_2, h_3, h_4 \in W_1$ such that $\{f, g, h_1, h_2, h_3, h_4\}$ is a basis for W_1 and $\{f, g, h_1\}$ is a basis for W_2.