Sample exam 3
Math 129b, Fall 2004

You will not be allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything which has been proven in class, in the homework, or in the reading.

1. (12 points) Let $T : V \to V$ be a linear operator. (Note that V may be infinite-dimensional.) Define what it means for $\lambda \in \mathbb{R}$ to be an eigenvalue of T.

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -5 \\ 0 & 2 & 0 \end{bmatrix} \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

It can be shown that $B = \{(7,0,3),(2,0,1),(-1,1,3)\}$ is a basis for $\mathbb{R}^3$ (i.e., you may take this as given). Find a product of matrices that is equal to the matrix $[T]_{B,B}$ (the matrix of T relative to the bases B, B).

3. (12 points) (T/F) There exists an onto linear function $S : \mathbb{R}^{33} \to \mathbb{R}^{26}$ such that $\operatorname{nullity} S = 8$.

4. (12 points) (T/F) It is possible that there exist linear functions $T : \mathbb{R}^3 \to \mathbb{P}_2$ and $L : \mathbb{P}_2 \to \mathbb{M}(2,4)$ such that $1 - x^2$ is an element of $\operatorname{im}(L \circ T)$ (the image of $L \circ T$).

5. (12 points) (T/F) Let $V = \mathbb{D}^\infty(\mathbb{R})$, the space of infinitely differentiable functions on $\mathbb{R}$, and let $S : V \to V$ be defined by $S(f(x)) = f(x^3)$. Then S is linear.

6. (12 points) (T/F) Let $T : \mathbb{M}(2,2) \to \mathbb{P}_4$ be a linear function such that $T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = x^2$, $T\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = x$, and $T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -2x - 3x^2$. It is possible that T is one-to-one.

7. (12 points) **PROOF QUESTION.** Let

$$V = \operatorname{span}\{1, \cos x, \sin x, e^x\} \subseteq \mathbb{F}(\mathbb{R}).$$

It can be shown that $B = \{1, \cos x, \sin x, e^x\}$ is a basis for V; i.e., you may take this as given. Furthermore, let $B' = \{1, x, x^2, x^3, x^4\}$ be the standard basis for $\mathbb{P}_4$, and let $T : V \to \mathbb{P}_4$ be a linear function such that $[T]_{B,B'}$, the matrix of T relative to B and B', is

$$[T]_{B,B'} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Prove that $x^3 - 2x^2 + 7$ is an element of $\operatorname{im} T$.

8. (18 points) **PROOF QUESTION.** Let V and W be vector spaces such that $\dim V = 5$ and $\dim W = 6$. Let U be a subspace of V such that $\dim U = 3$. Prove that there exists a linear function $T : V \to W$ such that $\ker T = U$.