Sample final exam
Math 129b, Fall 2004

You will not be allowed to use books, notes, or calculators. Unless otherwise stated, you may take as given anything which has been proven in class, in the homework, or in the reading.

1. (16 points) Let $T : V \to W$ be linear.

(a) Define $\ker T$. (I.e., define the kernel of T.)

(b) Define $\operatorname{im} T$. (I.e., define the image of T.)

2. (18 points) Let V be a vector space, let $T : V \to V$ be linear, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for V such that:

- $T(\mathbf{v}_1) = 6\mathbf{v}_2 - 3\mathbf{v}_1$;
- $\mathbf{v}_2$ is an eigenvector of T with eigenvalue -5; and
- $[T(\mathbf{v}_3)]_B = \begin{bmatrix} 2 \\ -7 \\ 11 \end{bmatrix}$, where $[T(\mathbf{v}_3)]_B$ is the coordinate vector of $T(\mathbf{v}_3)$ relative to the basis B.

Find $[T]_{B,B}$ (the matrix of T relative to the bases B, B), using the definition of $[T]_{B,B}$. No explanation necessary, but show all your work. (In particular, you should write down the definition of $[T]_{B,B}$ and show how your answer is obtained from the definition.)

3. (T/F) Let $T : \mathbb{M}(2,2) \to \mathbb{R}^5$ be linear and one-to-one. It must be the case that T is also onto.

4. (T/F) Let V be a vector space such that $\dim V = 6$. It must be the case that there exist $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

5. (T/F) Let A be a 4×4 matrix. It is possible that the characteristic polynomial of A is $(\lambda - 5)^2(\lambda + 1)(\lambda + 2)$ and $\dim E_A(4) = 1$, where $E_A(4)$ is the 4-eigenspace of A.

6. (T/F) Let V be a vector space. It is impossible for V to contain 0 vectors, and it is impossible for V to contain exactly 2 distinct vectors.

7. (T/F) Let W be a subspace of $\mathbb{R}^4$ such that $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for some $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$. It must be the case that $\dim W = 3$.

8. (T/F) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$. Then S is a subspace of $\mathbb{R}^3$.

9. (T/F) Let W be a subspace of $\mathbb{P}_4$ (the space of all polynomials of degree ≤ 4). It must be the case that some subset of $\{1, x, x^2, x^3, x^4\}$ is a basis for W.

10. (T/F) It is possible to find a continuous real-valued function f with domain $[0, 1]$ such that

$$\left(\int_0^1 x^2 f(x)\, dx\right)^2 > \frac{1}{5} \int_0^1 f(x)^2\, dx.$$

11. PROOF QUESTION. Let A be a fixed 2×2 matrix, and let

$$U = \left\{ X \in \mathbb{M}(2,2) \,\middle|\, AX = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

In other words, let U be the set of all 2×2 matrices X such that $AX = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Prove that U is a subspace of $\mathbb{M}(2,2)$.

12. PROOF QUESTION. Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $T : V \to V$ be a linear function with the property that, for all $\mathbf{v}, \mathbf{w} \in V$, we have

$$\langle T(\mathbf{v}), T(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle .$$

Prove that if λ is an eigenvalue of T, then either $\lambda = 1$ or $\lambda = -1$.

13. PROOF QUESTION. Let V be a vector space such that $\dim V = 5$, and let W be a finite-dimensional subspace of $\mathbb{F}(\mathbb{R})$ (the vector space of all real-valued functions with domain $\mathbb{R}$). Also, let f, g be elements of W such that

$$\begin{aligned} f(3) &= 2, & f(7) &= -1, \\ g(3) &= 0, & g(7) &= 5. \end{aligned}$$

Prove that there exists a linear function $T : V \to W$ such that $f \in \operatorname{im} T$, $g \in \operatorname{im} T$, and $\operatorname{nullity} T = 3$.

14. PROOF QUESTION. Let A be a 4×4 matrix with the following properties:

- A has the form $\begin{bmatrix} 5 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$, where the $*$ entries are unspecified real numbers;
- -3 is an eigenvalue of A; and
- There exist linearly independent vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$ such that $A\mathbf{u} = 7\mathbf{u}$ and $A\mathbf{v} = 7\mathbf{v}$.

Prove that there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.