Axioms for the real numbers: Field axioms

- (A1) For all $a, b \in \mathbb{R}$, $a + b = b + a$. (+ commutative)
- (A2) For all $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$. (+ associative)
- (A3) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $a + 0 = a$. (Zero element)
- (A4) For all $a \in \mathbb{R}$, there exists $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$. (Negatives)
- (M1) For all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$. ($\cdot$ commutative)
- (M2) For all $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. ($\cdot$ associative)
- (M3) There exists $1 \in \mathbb{R}$ s.t. for all $a \in \mathbb{R}$, $a \cdot 1 = a$. (Unit element)
- (M4) For all $a \neq 0$ in $\mathbb{R}$, there exists $(1/a) \in \mathbb{R}$ such that $a \cdot (1/a) = 1$. (Reciprocals)
- (D) For all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$. (Distributive)

This all works if we replace $\mathbb{R}$ with rationals, complex numbers, integers mod 2 (the set $\{0, 1\}$, taking $1 + 1 = 0$).

Axioms for the real numbers: Order axioms

There exists a subset P of $\mathbb{R}$ called the *positive* numbers. Instead of $a \in P$, we write $a > 0$.
The positive numbers satisfy:

(O1) For all $a, b \in \mathbb{R}$, if $a > 0$ and $b > 0$, then $a + b > 0$.

(O2) For all $a, b \in \mathbb{R}$, if $a > 0$ and $b > 0$, then $ab > 0$.

(O3) For any a, exactly one of $a > 0$, $a = 0$, or $-a > 0$ holds.

Numbers such that $-a > 0$ are called *negative* numbers, and we also write $a < 0$. We write $a < b$ to mean that $b - a > 0$.
This all works if we replace $\mathbb{R}$ with rationals.

Axioms for the real numbers: Completeness

(C) Every nonempty set of real numbers that has an upper bound also has a **least** upper bound (supremum).

It can be shown that the axioms (A1)–(A4), (M1)–(M4), (D), (O1)–(O3), and (C) determine $\mathbb{R}$ completely; that is, any other object with the same properties must be essentially the same as $\mathbb{R}$.