Summary of epsilonics
Math 131A

In this course, we have seen three key definitions that begin, “For every $\epsilon > 0$,” namely: the limit of a sequence, the limit of a function, and the continuity of a function. These definitions are compared in the following table, in which we assume (x_n) is a sequence, $f : A \to \mathbf{R}$ is a function, and c is a cluster point of A.

Idea	Epsilonic definition	Sequential criterion
$\lim_{n\to\infty} x_n = L$	For every $\epsilon > 0$, there exists a $K(\epsilon) \in \mathbf{N}$ such that if $n \in \mathbf{N}$, $n \geq K(\epsilon)$, then $\lvert x_n - L\rvert < \epsilon$.	n/a
$\lim_{x\to c} f(x) = L$	For every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $x \in A$, $\lvert x - c\rvert < \delta(\epsilon)$, $x \neq c$, then $\lvert f(x) - L\rvert < \epsilon$.	If (x_n) is a sequence in A such that $\lim_{n\to\infty} x_n = c$ and for all $n \in \mathbf{N}$, $x_n \neq c$, then $\lim_{n\to\infty} f(x_n) = L$.
f continuous at c	For every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $x \in A$, $\lvert x - c\rvert < \delta(\epsilon)$, then $\lvert f(x) - f(c)\rvert < \epsilon$.	If (x_n) is a sequence in A such that $\lim_{n\to\infty} x_n = c$, then $\lim_{n\to\infty} f(x_n) = f(c)$.

There is a fourth key definition, that of differentiability, that could be expressed directly in epsilonic terms, but is perhaps more memorably defined using the definition of limit:

Definition. Let I be an interval, let $f : I \to \mathbf{R}$ be a function, and let c be a point of I. To say that f is differentiable at c means that $\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$ exists. If f is differentiable at c, we define $f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$.