Math 131A, problem set 03
Outline due: Wed Sep 10
Completed version due: Mon Sep 15
Last revision due: Mon Oct 20

All problem numbers refer to Bartle and Sherbert.
Problems to be done but not turned in: (2.4) 1, 2, 4, 5, 7, 11, 14, 17; (2.5) 2, 10.
Problems to be turned in:

1. Let S be a bounded nonempty subset of $\mathbf{R}$, and let u be a real number. Prove that the following statements are equivalent:

   (i) If $z \in \mathbf{R}$, $z < u$, then z is not an upper bound of S.

   (ii) If $z \in \mathbf{R}$, $z < u$, then there exists $s_z \in S$ such that $z < s_z$.

   (I.e., prove that (i) is true if and only if (ii) is true.)

2. Let
$$S = \left\{ \frac{n}{n+m} \,\middle|\, m, n \in \mathbf{N} \right\}.$$
   Find $\inf S$ and $\sup S$, and prove your answers.

3. Let $f : D \to \mathbf{R}$ and $g : D \to \mathbf{R}$ be functions, and suppose that $f(x) \leq g(y)$ for all $x, y \in D$. Prove that $\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y)$.

4. (2.4) 18.

5. Let A and B be nonempty bounded subsets of $\mathbf{R}$ such that $a > 0$ for all $a \in A$ and $b > 0$ for all $b \in B$, and let $C = \{ab \mid a \in A, b \in B\}$. Prove that C is bounded, and that $\sup C = (\sup A)(\sup B)$. (Suggestion: Use the Arbitrarily Close Criterion.)

6. Review the definition of $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ (pp. 3–4 of Bartle and Sherbert).

   Let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ be a sequence of nested bounded **open** intervals. Must $\bigcap_{n=1}^{\infty} I_n$ be nonempty? Either prove that $\bigcap_{n=1}^{\infty} I_n$ is always nonempty, or demonstrate a specific counterexample $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, and prove that $\bigcap_{n=1}^{\infty} I_n = \emptyset$ in your example.