Math 131A, problem set 06
Outline due: Wed Oct 08
Completed version due: Mon Oct 13
Last revision due: Wed Nov 19

All problem numbers refer to Bartle and Sherbert.
Problems to be done but not turned in: (3.4) 1, 7, 8, 12, 13, 14; (3.5) 1, 3, 5, 8.
Problems to be turned in:

1. Let (x_n) be a convergent sequence, and let (y_n) be a divergent sequence. Prove that $(x_n + y_n)$ is divergent.

2. Let $[a, b]$ be a nonempty closed interval in $\mathbf{R}$. Prove that if (x_n) is a sequence such that $x_n \in [a, b]$ for all $n \in \mathbf{N}$, then (x_n) has a subsequence that converges to some $c \in [a, b]$. (Compare problem 3b.)

3. (a) Give an example of a sequence (x_n) in $\mathbf{R}$ such that no subsequence of (x_n) converges to a limit in $\mathbf{R}$. Prove your answer.
   (b) Give an example of a sequence (x_n) in the open interval $(0, 1)$ such that no subsequence of (x_n) converges to a limit in $(0, 1)$. Prove your answer.

4. For a subset S of $\mathbf{R}$, we define a *limit point* of S to be some $u \in \mathbf{R}$ such that, for any $\epsilon > 0$, there exists some $s \in S$, $s \neq u$, such that $|s - u| < \epsilon$.

   Let $[a, b]$ be a nonempty closed interval in $\mathbf{R}$, and let S be an infinite subset of $[a, b]$. Prove that S has some limit point $u \in [a, b]$. (Suggestions: Use Bolzano-Weierstrass. Don't forget the condition $s \neq u$ in the definition of limit point.)

5. Let (x_n) be a bounded sequence, and for each $n \in \mathbf{N}$, let

$$T_n = \{x_k \mid k \geq n\},$$
$$s_n = \sup T_n = \sup \{x_k \mid k \geq n\}.$$

   Recall that in PS05, you showed that (s_n) is decreasing and convergent. Let $S = \lim(s_n)$. Prove that there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty}(x_{n_k}) = S$. (As a first step, write out a clear procedure for choosing n_1, n_2, and so on. Note that one tricky aspect of choosing a subsequence (x_{n_k}) is ensuring that $n_1 < n_2 < n_3 < \ldots$.)

6. Suppose that (x_n) and (y_n) are Cauchy sequences. Use the definition of Cauchy sequence to prove that $(x_n + y_n)$ is Cauchy.