**Math 131A, problem set 11**
**Outline due: Mon Nov 24**
**Completed version due: Mon Dec 01**
**Last revision due: TBA**

All problem numbers refer to Bartle and Sherbert.

**Problems to be done but not turned in:** (5.3) 11, 14, 15, 16, 17, 19; (5.4) 1, 3, 6, 8.

**Problems to be turned in:**

1. The goal of this problem is to prove a special case of the two-dimensional version of the Maximum-Minimum Theorem. We first extend our basic definitions to two dimensions as follows.

   - A **sequence** in $\mathbf{R}^2$ is defined to be a function $X : \mathbf{N} \to \mathbf{R}^2$. Writing $X(n) = (x_n, y_n)$, we see that a sequence in $\mathbf{R}^2$ can also be thought of as a pair of sequences in $\mathbf{R}$, one to determine the $x$-coordinate of the sequence, and one to determine the $y$-coordinate.
   - If $n_1 < n_2 < n_3 < \dots$ is a strictly increasing sequence of natural numbers, then the sequence $(x_{n_k}, y_{n_k})$ (in the variable $k$) is said to be a **subsequence** of $(x_n, y_n)$.
   - We say that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ if $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$. We also use the terms **convergent** and **divergent** as we do with sequences in $\mathbf{R}$.
   - Let $A$ be a subset of $\mathbf{R}^2$, and let $f : A \to \mathbf{R}$ be a real-valued function on $A$. To say that $f$ is **continuous** at the point $(c, d)$ means that if $\lim_{n\to\infty}(x_n, y_n) = (c, d)$, then $\lim_{n\to\infty}(f(x_n, y_n)) = f(c, d)$. (There is also an equivalent $\epsilon$-$\delta$ version of the definition of continuity in two dimensions, but the sequential version is what we need.)
   - For $A \subseteq \mathbf{R}^2$ and $f : A \to \mathbf{R}$, we say that is **continuous on** $A$ if $f$ is continuous at every $(c, d) \in A$, and we say that $f$ is **bounded on** $A$ if there exists some $M > 0$ such that for all $(x, y) \in A$, $|f(x, y)| \leq M$. We also say that $f$ **has an absolute maximum** (resp. **minimum**) **on** $A$ if there exists $(x^*, y^*) \in A$ (resp. $(x_*, y_*) \in A$) such that for all $(x, y) \in A$, we have $f(x, y) \leq f(x^*, y^*)$ (resp. $f(x, y) \leq f(x_*, y_*)$).

   Let $D = [a, b] \times [c, d]$ be a closed and bounded rectangle in $\mathbf{R}^2$.

   (a) Prove that any sequence $(x_n, y_n)$ in the rectangle $D$ has a convergent subsequence.

   (b) Let $f : D \to \mathbf{R}$ be a continuous function on the rectangle $D$. Prove that $f$ is bounded on $D$.

   (c) Let $f : D \to \mathbf{R}$ be a continuous function on the rectangle $D$. Prove that $f$ has both an absolute maximum and an absolute minimum on $D$.

**Note:** This theorem is actually true for any **closed and bounded** subset $D$ of $\mathbf{R}^2$ (or $\mathbf{R}^n$, for that matter). See sections 11.3–11.4 of Bartle and Sherbert for a proof. (Or take Math 175!)

2. (a) Does there exist a continuous **onto** function $f : (0,1) \to [0,1]$? Give an example or prove that no such function exists.

   (b) Does there exist a continuous **onto** function $f : [0,1] \to (0,1)$? Give an example or prove that no such function exists.

3. (a) Let $f : [0,1] \to \mathbf{R}$ and $g : [0,1] \to \mathbf{R}$ be continuous functions such that $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$. Prove that there exists some $x \in [0,1]$ such that $f(x) = g(x)$.

   (b) On Tuesday, November 11, Frances and Genji are both on Mt. Shasta. Frances begins her day by departing from the base of the mountain at 5am, and ends the day at 7pm at a camp at the summit of the mountain. Genji wakes up at 8am at his camp at the summit, takes his time climbing down, and reaches the base of the mountain at 4pm.

   Prove that at some point in time on November 11, Frances and Genji are at exactly the same elevation. You may make reasonable assumptions about continuity, but clearly state what those assumptions are.

4. (5.4) 2.

5. (5.4) 5.

6. (5.4) 10.