Math 131A, problem set 12
CORRECTED DEC 9, 10:30pm
Outline due: Wed Dec 03
Completed version due: Mon Dec 08
Last revision due: TBA

All problem numbers refer to Bartle and Sherbert.

Problems to be done but not turned in: (6.1) 5, 6, 8, 10, 11, 16; (6.2) 6, 9, 10, 17, 20(a,b).

Problems to be turned in:

1. Let r be a rational number such that x^r is defined for all $x \in \mathbf{R}$. (In other words, suppose $r = m/n$, where it is not the case that m is odd and n is even.) Let $g_r : \mathbf{R} \to \mathbf{R}$ be defined by
$$g_r(x) = \begin{cases} x^r & \text{for } x \in \mathbf{Q}, \\ 0 & \text{for } x \notin \mathbf{Q}. \end{cases}$$
Some useful facts:
   - $a^0 = 1$ for all $a \in \mathbf{R}$ (including $a = 0$);
   - If $t > 0$ and $0 < a < b$, then $0 < a^t < b^t$; and
   - If $t < 0$ and $0 < a < b$, then $0 < b^t < a^t$.

   (a) Prove that for $r > 0$, g_r is continuous at 0.
   (b) Prove that for $r > 1$, g_r is differentiable at 0.
   (c) Prove that for $0 < r \leq 1$, g_r is not differentiable at 0.

2. (6.1) 2.

3. DELETED

4. Let I be an interval, let $f : I \to \mathbf{R}$ be a function, and let c be a point in I. Prove that the following are equivalent:
   - f is differentiable at c.
   - There exists some $m \in \mathbf{R}$ such that the function
   $$E(x) = f(x) - f(c) - m(x - c)$$
   satisfies the condition $\lim_{x \to c} \dfrac{E(x)}{x - c} = 0$.

   Furthermore, prove that if these conditions hold, then $m = f'(c)$.

   (Note: The function $E(x)$ can be thought of as the error in the linear approximation $f(x) \approx f(c) + m(x - c)$. Differentiability says that the error $E(x)$ approaches 0 as $x \to c$ faster than any linear function of $x - c$.)

5. (6.2) 13.

6. Suppose $f : [0, \infty) \to \mathbf{R}$ is a function such that $f''(x) > 0$ for $x \in [0, \infty)$, $f(0) = 0$, and $f'(0) > 0$. Use the results of 6.1–6.2 to prove that there exists some $m > 0$ such that $f(x) \geq mx$ for all $x \in [0, \infty)$. (Make sure you carefully establish the existence and continuity of all functions that you need to exist or to be continuous.)

7. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and suppose that $f''(x)$ exists for all $x \in (a, b)$. Furthermore, suppose that the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of $y = f(x)$ at a third point $(c, f(c))$, with $a < c < b$. Use the results of 6.1–6.2 to prove that for some $t \in (a, b)$, $f''(t) = 0$.