The ratio test for positive series
Math 131A

Theorem. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series such that* $a_n > 0$ *for all* $n \in \mathbf{N}$*, and suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then:

1. *If* $L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

2. *If* $L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Suppose $L < 1$. Let $\epsilon = \dfrac{1-L}{2} > 0$, and let $r = L + \epsilon = \dfrac{L+1}{2} < 1$. By the definition of $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, there exists some natural number $K = K(\epsilon)$ such that if $n \in \mathbf{N}$, $n \geq K$, then $\left|\dfrac{a_{n+1}}{a_n} - L\right| < \epsilon$. Therefore, for $n \geq K$, we have

$$\frac{a_{n+1}}{a_n} < L + \epsilon = r.$$

Since $a_{n+1} < a_n r$ for $n \geq K$, an easy induction then shows that for $m \geq 1$, $a_{K+m} < a_K r^m$. It follows that the series $\sum_{n=K+1}^{\infty} a_n = \sum_{m=1}^{\infty} a_{K+m}$ converges by comparison with the geometric series $\sum_{m=1}^{\infty} a_K r^m$ $(r < 1)$. Then, since finitely many terms do not affect the convergence of a series (see PS07), $\sum_{n=1}^{\infty} a_n$ converges as well.

Suppose $L > 1$. Let $\epsilon = \dfrac{L-1}{2} > 0$, and let $r = L - \epsilon = \dfrac{L+1}{2} > 1$. By the definition of $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, there exists some natural number $K = K(\epsilon)$ such that if $n \in \mathbf{N}$, $n \geq K$, then $\left|\dfrac{a_{n+1}}{a_n} - L\right| < \epsilon$. Therefore, for $n \geq K$, we have

$$\frac{a_{n+1}}{a_n} > L - \epsilon = r.$$

Since $a_{n+1} > a_n r$ for $n \geq K$, an easy induction then shows that for $m \geq 1$,

$$a_{K+m} > a_K r^m.$$

By an argument similar to the $L < 1$ case, the series $\sum_{n=K+1}^{\infty} a_n = \sum_{m=1}^{\infty} a_{K+m}$ diverges by comparison with the geometric series $\sum_{m=1}^{\infty} a_K r^m$ $(r > 1)$, so $\sum_{n=1}^{\infty} a_n$ diverges as well. $\square$