Sample Final Exam
Math 131A, Fall 2008

1. (20 points) Let A be a nonempty subset of $\mathbb{R}$. Define what it means to be an upper bound of A, define what it means to be the supremum of A, and state the Completeness Axiom.

2. (16 points) State the Maximum-Minimum Theorem.

For questions 3–8, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (13 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x \to 5} f(x) = 13$. Then it must be the case that f is differentiable at $x = 5$, i.e., it must be the case that $f'(5)$ exists.

4. (13 points) Let $f : (0, 2) \to \mathbb{R}$ be a bounded and continuous function on $(0, 2)$. Then it must be the case that that there exists $x \in (0, 2)$ such that $f(x) = \sup \{f(x) \mid x \in (0, 2)\}$.

5. (13 points) Let X and Y be sequences. If XY converges, then it must be the case that both X and Y converge.

6. (13 points) Let the sequence (x_n) be defined by $x_n = 2 \sin(n^2 + 7n)$. Then (x_n) has a convergent subsequence.

7. (13 points) Let $A = \mathbb{R} \setminus \{0\}$, and let $f : A \to \mathbb{R}$ be a function such that $|f(x)| \leq 2$ for all $x \in A$. Then it must be the case that $\lim_{x \to 0} f(x)$ exists.

8. (13 points) Let $\sum_{n=1}^{\infty} a_n$ be a series, and let (s_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$. If $s_n \geq s_{n+1}$ for $n \geq 300$, and $\sum_{k=1}^{n} a_k = a_1 + \cdots + a_n > 2$ for all $n \in \mathbb{N}$, then it must be the case that $\sum_{n=1}^{\infty} a_n$ converges.

9. (16 points) **PROOF QUESTION.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$f(x) = \sqrt{|x|} = \begin{cases} \sqrt{x} & \text{if } x \geq 0, \\ \sqrt{-x} & \text{if } x < 0. \end{cases}$$

Prove that f is not differentiable at 0.

10. (16 points) **PROOF QUESTION.** Let $A = \mathbb{R} \setminus \{-2\}$, and let $f : A \to \mathbb{R}$ be given by the formula

$$f(x) = \frac{x + 7}{x + 2}.$$

Use either the epsilon-delta definition of continuity or the Sequential Criterion for Continuity to prove that f is continuous at $c = 3$.

11. (18 points) **PROOF QUESTION.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at every $c \in \mathbb{R}$, suppose that

$$-2 \leq f'(x) \leq 3 \qquad\qquad \text{for all } x \in [1, 14],$$

and suppose that $f(1) = 10$. What is the smallest possible M such that we can be sure that $f(14) \leq M$? In other words, what is the largest possible value for $f(14)$? Use the Mean Value Theorem to **prove** that $f(14) \leq M$ for your choice of M.

12. (18 points) **PROOF QUESTION.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} g(x) = -4$. Suppose that (x_n) and (y_n) are sequences such that $\lim\limits_{n \to \infty} x_n = 1$, $x_n \neq 1$ for all $n \in \mathbb{N}$, and $f(x_n) \leq y_n \leq g(x_n)$ for all $n \in \mathbb{N}$. Prove that $\lim\limits_{n \to \infty} y_n = -4$.

13. (18 points) **PROOF QUESTION.** Let $f : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be functions that are continuous on $\mathbb{R}$, and suppose that $f(1) = 3$ and $g(1) = 4$. Prove that there exists a δ-neighborhood $V_\delta(1)$ such that if $x \in V_\delta(1)$, then $f(x) + g(x) > 5$.