**Math 31**
**A brief guide to comparisons of various types**

**Factorials.** To review, the symbol $n!$ is defined to be

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1,$$

with the additional definition that $0! = 1$. For example, $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$.

**Comparisons of different kinds of functions.** The key point to remember is that for $t > 0$, $a > 1$:

$$1 << \ln n << n^t << a^n << n!$$

Here "$f(n) << g(n)$" means that as $n \to \infty$, $\dfrac{f(n)}{g(n)} \to 0$, or in other words,

$$f(n) << g(n) \qquad \text{means} \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

For example, since $\ln n << n^{1/2} = \sqrt{n}$, we have that

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0,$$

and since $10^n << n!$, we have that

$$\lim_{n\to\infty} \frac{10^n}{n!} = 0.$$

**Comparisons of fractions.** The key points to remember are:

- A larger numerator makes a larger fraction, and a smaller numerator makes a smaller fraction.
- A larger denominator makes a *smaller* fraction, and a smaller denominator makes a *larger* fraction.

For example,

$$\begin{aligned} \frac{n^2+1}{2n^4+5} &\le \frac{n^2+1}{2n^4} && \text{(smaller denominator)} \\ &\le \frac{n^2+n^2}{2n^4} && \text{(larger numerator)} \\ &= \frac{2n^2}{2n^4} = \frac{1}{n^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{n^2+1}{2n^3+5} &\ge \frac{n^2}{2n^3+5} && \text{(smaller numerator)} \\ &\ge \frac{n^2}{2n^3+5n^3} && \text{(larger denominator)} \\ &= \frac{n^2}{7n^3} = \frac{1}{7n}. \end{aligned}$$

**Math 31**
**How much justification is needed for convergence/divergence?**

When you are asked to determine if a series converges or diverges, and then justify your answer, here are some guideline for how much detail your answer should contain.

The basic principle is:

Whenever you say a series converges or diverges, you have to say why.

More precisely, you need to say what test or principle ensures that the series converges or diverges. For example:

**Question.** Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

**Model answer.** Notice that $\frac{1}{n^2+1} < \frac{1}{n^2}$, since a smaller denominator produces a bigger fraction. Also, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since it is a $p$-series with $p = 2 > 1$. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, by the comparison test.

Really, the model answer isn't that long, but yes, you really *are* supposed to write down all of that stuff. Of course, in an exam situation, you can abbreviate:

**Shorter model answer.**

$\frac{1}{n^2+1} < \frac{1}{n^2}$ (smaller denominator $\Rightarrow$ bigger fraction)

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p$-series with $p = 2 > 1$)

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges (comparison test)

But all of that information needs to be there.