Sample Final Exam, Math 32, Fall 2006

The following questions are included only to indicate the style of questions that will appear on the final exam. You will be given the following definite integrals:

$$\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = \pi,$$

$$\int_0^{\pi/2} \sin^2 x\,dx = \int_{\pi/2}^{\pi} \sin^2 x\,dx = \int_{\pi}^{3\pi/2} \sin^2 x\,dx = \int_{3\pi/2}^{2\pi} \sin^2 x\,dx = \frac{\pi}{4},$$

$$\int_0^{\pi/2} \cos^2 x\,dx = \int_{\pi/2}^{\pi} \cos^2 x\,dx = \int_{\pi}^{3\pi/2} \cos^2 x\,dx = \int_{3\pi/2}^{2\pi} \cos^2 x\,dx = \frac{\pi}{4}.$$

1. Consider the following parametric equations:

$$x = 2\sin t, \qquad y = 4\sin^2 t.$$

Eliminate the parameter to find a Cartesian equation of the corresponding curve, sketch the curve, and in **ONE SENTENCE**, describe the motion of a particle with position (x, y) as t varies continuously from 0 to 2π.

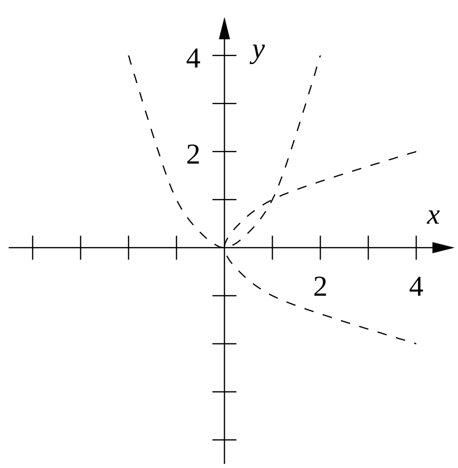
2. Convert the point $(\rho, \theta, \varphi) = (3, \pi/3, 5\pi/6)$ from spherical coordinates to rectangular coordinates, and graph the point.

3. Let $\mathbf{a} = \langle 2, 4, -1\rangle$, $\mathbf{b} = \langle 3, 0, 7\rangle$. Calculate $\mathbf{a}+\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$.

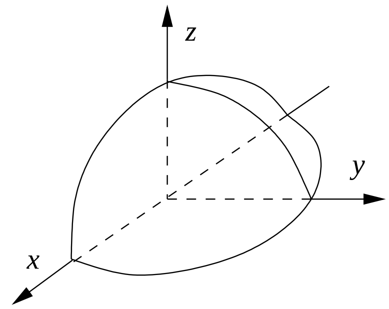
4. For $0 \le t \le 2$, the path with parametric equations

$$x = -t, \qquad y = t^2,$$

travels along some part of one of the two dashed-line curves shown below. Fill in the part of the dashed-line curve that matches the above parametric equations with $0 \le t \le 2$, and draw arrows on the part you fill in to indicate the direction of motion of the path. Show all your work.



5. Let E be the portion of the upper hemisphere of the sphere of radius 7 centered at the origin with $y \geq 0$, as sketched in the picture below. In other words, E is the quarter-sphere of radius 7 with $y \geq 0$ and $z \geq 0$.

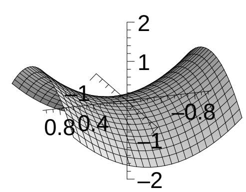


Set up, but **DO NOT EVALUATE**, the triple integral $\iiint_E x^4 y^2 z \, dV$ as an iterated integral in spherical coordinates.

6. Let $f(x, y) = y \sin(xy) - y^2$. Find an equation for the tangent plane to $z = f(x, y)$ at $(x, y) = (\pi, 2)$. No explanation necessary. **DO NOT SIMPLIFY** your answer.

7. Let P be the point $(5, -3, 2)$ and let Q be the point $(4, 0, 3)$. Find the equation of the plane that passes through the point P and is perpendicular to the vector $\overrightarrow{PQ}$ (i.e., perpendicular to the vector from the point P to the point Q). No explanation necessary, but show all your work.

8. Which of the following equations $z = f(x, y)$ best matches the graph shown below?



(a) $z = x^2 + y^2$

(b) $z = x^2 - y^2$

(c) $z = -x^2 + y^2$

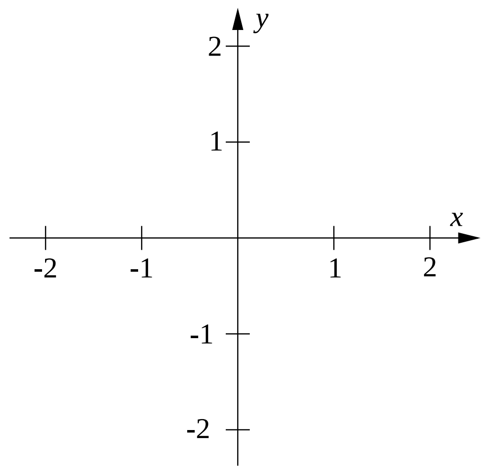
(d) $z = -x^2 - y^2$

Circle the correct equation, and briefly **EXPLAIN** your answer. (In particular, explain why the other equations do not match as well as your choice.)

9. Consider the double integral

$$\int_{-1}^{1}\int_{x^2}^{1} f(x,y)\,dy\,dx.$$

(a) Draw the region of integration on the following axes.



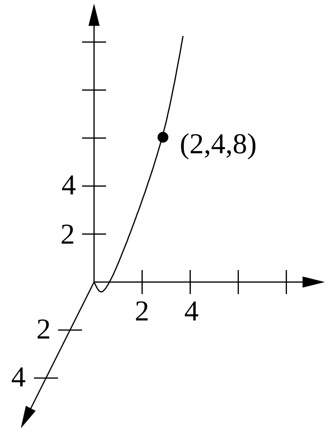
(b) Change the order of integration, i.e., express this integral as an iterated integral, integrating the variable x first. **DO NOT EVALUATE THIS INTEGRAL.**

10. Suppose that, for $t \geq 0$, a particle travels along the path $\mathbf{r}(t)$ with vector equation

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle .$$

(a) Find $\mathbf{v}(2)$, the velocity of the particle at time $t = 2$. No explanation necessary, but show all your work.

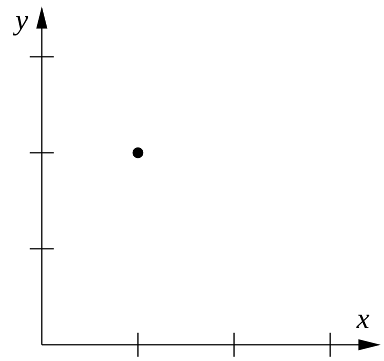
(b) The path $\mathbf{r}(t)$ is sketched on the axes below, and the indicated point is $\mathbf{r}(2) = \langle 2, 4, 8 \rangle$. **Draw** the vector $\mathbf{v}(2)$ based at the point $(2, 4, 8)$, and briefly **EXPLAIN** the relationship between the direction of the vector $\mathbf{v}(2)$ that you just drew and the space curve $\mathbf{r}(t)$, **as precisely as possible**.



11. A bug is crawling around the xy-plane, and is currently located at $(1, 2)$. The temperature $T(x, y)$ at the point (x, y) is $T(x, y) = 78 + 10e^{-x^2-y^2}$.

(a) Calculate $\nabla T(x, y)$, the gradient of T at the point (x, y).

(b) Suppose the bug now wants to get warmer as quickly as possible. In which direction should the bug travel away from $(1, 2)$? Indicate your answer on the axes below, and briefly **JUSTIFY** your answer.



12. Let D be a flat washer-shaped plate in the xy-plane that is bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and suppose that the density of D at (x, y) is $\rho(x, y) = x^2 + 2$. Find the mass of the flat plate D. No explanation necessary, but show all your work.

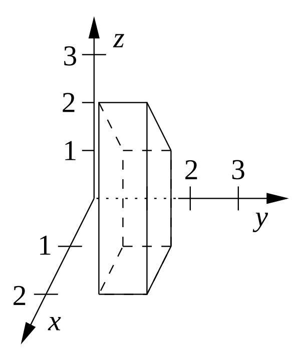
13. Let $f(x, y)$ be a function such that

$$f_x(x, y) = x - y, \qquad f_y(x, y) = y^2 - x - 2.$$

(a) Find all critical points of $f(x, y)$. (Hint: There are two critical points, and one of them is $(2, 2)$.) Show all your work.

(b) Classify each critical point of $f(x, y)$ as a local minimum, local maximum, or saddle point. Show all your work.

14. Let E be the three-dimensional region described by $1 \le x \le 2$, $2 \le y \le 3$, $0 \le z \le 2x$, as shown in the following picture.



Calculate the triple integral $\iiint_E y^2 \sin(x^2 + 1)\, dV$. Show all your work, and **DO NOT SIMPLIFY** your final numerical answer.