# ASCENDING HNN EXTENSIONS OF POLYCYCLIC GROUPS ARE RESIDUALLY FINITE

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Abstract. We prove that every ascending HNN extension of a polycyclic-by-finite group is residually finite. We also give a criterion for the residual finiteness of an ascending HNN extension of a residually nilpotent group, and apply this criterion to recover a result of Sapir on the residual finiteness of certain ascending HNN extensions of free groups.

## 1. Introduction

Let $P$ be a group, and let $\varphi\colon P \to P$ be a monomorphism. The *ascending HNN extension*, or *mapping torus*, corresponding to $\varphi$ is the group

$$P_\varphi = \langle P, t \,|\, p^t = \varphi(p) \rangle, \tag{1}$$

where $x^y$ denotes $y^{-1}xy$.

Ascending HNN extensions are an interesting and well-studied class of groups. For example, Feighn and Handel [FH99] have recently shown that ascending HNN extensions of free groups are *coherent* (that is, their f.g. subgroups are finitely presented).

Recall that a group $G$ is *residually finite* if each nontrivial element of $G$ has a nontrivial image in some finite quotient of $G$. Now, the normal forms of an ascending HNN extension behave much like the normal forms of a split extension, and as observed by Malćev, a split extension $P \rtimes_\varphi \mathbb{Z}$ of a f.g. residually finite group $P$ is residually finite [Mil71, III.A, Thm. 7]. One might therefore hope that an ascending HNN extension of a f.g. residually finite group is also residually finite. However, there exist examples of ascending HNN extensions $P_\varphi$ of f.g. residually finite groups $P$ where $P_\varphi$ has very few finite quotients, or where $P_\varphi$ even fails to be Hopfian [SW]. It is therefore interesting to see how the nature of the base group governs the residual finiteness of the ascending HNN extension.

In particular, because polycyclic groups (see Section 3) are among the most residually finite of infinite groups, one might hope that their ascending HNN extensions are well-behaved. This class of groups is known to have many interesting properties; most notably, the class of ascending HNN extensions of polycyclic groups is precisely the class of f.g. coherent solvable groups ([BS79], [Gro78]).

Our main result is:

**Theorem 1.1.** *Let $\varphi\colon P \to P$ be a monomorphism of a polycyclic-by-finite group. Then $P_\varphi$ is residually finite.*

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Note that several cases of Theorem 1.1 are already known. Specifically, when $P$ is f.g. and abelian, $P_\varphi$ is a f.g. metabelian group, and therefore residually finite by a theorem of P. Hall (see [Hal49] or [Rob96, 15.4.1]); and when $P$ is a f.g. free nilpotent group, $P_\varphi$ is residually finite by a theorem of Moldavanskiĭ [Mol92].

We now give a brief overview of this paper. In Section 2 and 3, we summarize some necessary background material, first from general group theory, and then from the theory of nilpotent and polycyclic groups. The main content of the paper begins in Section 4, with a characterization of f.g. groups whose ascending HNN extensions are all residually finite. Next, in Section 5, we show that ascending HNN extensions of torsion-free nilpotent groups are residually finite in a particularly strong sense, and in Section 6, we show that the class of groups whose ascending HNN extensions are residually finite is closed under certain extensions. We then combine the results of Sections 4–6 to prove Theorem 1.1 in Section 7. Finally, in Section 8, we provide a criterion for the residual finiteness of ascending HNN extensions of f.g. residually nilpotent groups, and apply this criterion to certain ascending HNN extensions of free groups and right-angled Artin groups.

## 2. General background

In this section, for the sake of completeness, we establish some conventions and present proofs of some well-known facts in group theory. The first is a slight extension of a theorem of Marshall Hall (see [LS77, Theorem 4.5]).

**Definition 2.1.** A subgroup $H$ of a group $G$ is *fully invariant* if $\varphi(H) \subseteq H$ for every endomorphism $\varphi$ of $G$.

**Lemma 2.2.** *Let $G$ be a group, let $n$ be a natural number, and let $N$ be the intersection of all subgroups of $G$ of index $\leq n$. Then $N$ is fully invariant, and if $G$ is f.g., then $N$ has finite index in $G$.*

Note that as a consequence, any finite index subgroup of a f.g. group $G$ contains a fully invariant finite index subgroup of $G$.

*Proof.* Let $\varphi$ be an endomorphism of $G$, let $N = \bigcap_{[G:H]\leq n} H$, and let

$$\varphi^{-1}(N) = \varphi^{-1}\left(\bigcap_{[G:H]\leq n} H\right) = \bigcap_{[G:H]\leq n} \varphi^{-1}(H). \tag{2}$$

Now, since the preimage $\varphi^{-1}(H)$ of a subgroup $H$ of index $\leq n$ also has index $\leq n$, we see that $\varphi^{-1}(N)$ is itself the intersection of subgroups of index $\leq n$ (though not necessarily all subgroups of index $\leq n$). It follows that $N \subset \varphi^{-1}(N)$, or in other words, that $\varphi(N) \subset N$. Furthermore, if $G$ is f.g., there are only finitely many homomorphisms $G \to S_k$ for $k \leq n$, and therefore, only finitely many subgroups of index $\leq n$. In that case, since the intersection of two finite index subgroups has finite index, we see that $N$ has finite index. □

**Lemma 2.3.** *Let $H$ be a subgroup of a group $G$, and suppose the diagram*

$$\begin{array}{ccccccccccc} 1 & = & G_0 & \lhd & G_1 & \lhd & \cdots & \lhd & G_n & = & G \\ & & \cup & & \cup & & & & \cup & & \\ 1 & = & H_0 & \lhd & H_1 & \lhd & \cdots & \lhd & H_n & = & H \end{array} \tag{3}$$

*commutes, where $H_i = H \cap G_i$, each row is a subnormal series, and each vertical map is an inclusion. Then each factor $H_i/H_{i-1}$ is naturally isomorphic to the subgroup $H_iG_{i-1}/G_{i-1}$ of the corresponding factor $G_i/G_{i-1}$, and*

$$[G : H] = \prod_{i=1}^{n}[(G_i/G_{i-1}) : (H_iG_{i-1}/G_{i-1})] = \prod_{i=1}^{n}[G_i : H_iG_{i-1}]. \tag{4}$$

*In particular, if $H_i/H_{i-1}$ has finite index in $G_i/G_{i-1}$ for $1 \leq i \leq n$, then $[G : H]$ is finite.*

*Proof.* By the second and third isomorphism theorems, we see that $H_i/H_{i-1} \cong H_iG_{i-1}/G_{i-1}$ and $[(G_i/G_{i-1}) : (H_iG_{i-1}/G_{i-1})] = [G_i : H_iG_{i-1}]$. Therefore, by an easy induction, it suffices to show that for $1 \leq i \leq n$,

$$[G_i : H_i] = [G_i : H_iG_{i-1}][G_{i-1} : H_{i-1}]. \tag{5}$$

However, we observe that $[H_iG_{i-1} : H_i]$ is the number of (right) cosets of $H_i$ in $H_iG_{i-1}$, which is the size of the orbit of $H_i$ under the (right) action of $G_{i-1}$ on the cosets of $H_i$, which is precisely $[G_{i-1} : G_{i-1} \cap H_i]$. Therefore,

$$\begin{aligned}[G_i : H_i] &= [G_i : H_iG_{i-1}][H_iG_{i-1} : H_i] \\ &= [G_i : H_iG_{i-1}][G_{i-1} : G_{i-1} \cap H_i] \\ &= [G_i : H_iG_{i-1}][G_{i-1} : H_{i-1}],\end{aligned} \tag{6}$$

and the lemma follows. □

**Lemma 2.4.** *Let $\varphi$ be an endomorphism of a group $G$, let $K$ be a $\varphi$-invariant normal subgroup of $G$, and let $\overline{\varphi}\colon (G/K) \to (G/K)$ be the map induced by $\varphi$. If both $\overline{\varphi}$ and $\varphi$ restricted to $K$ are isomorphisms, then $\varphi$ is an isomorphism.*

*Proof.* For $x \in \ker\varphi$, since $\overline{\varphi}$ is injective, we must have $x \in K$, and since $\varphi$ restricted to $K$ is injective, we must have $x = 1$. For $y \in G$, since $\overline{\varphi}$ is surjective, there exists some $x_0 \in G$ such that $\varphi(x_0) = yk$ for some $k \in K$, and since $\varphi$ restricted to $K$ is surjective, there exists some $x_1 \in K$ such that $\varphi(x_1) = k^{-1}$, in which case $\varphi(x_0x_1) = y$. □

**Definition 2.5.** Let $G$ be a group. For $x, y \in G$, we define $[x, y] = xyx^{-1}y^{-1}$, and for subgroups $H, K$ of $G$, we define $[H, K] = \langle [x, y],\ x \in H,\ y \in K\rangle$.

**Definition 2.6.** For a group $G$ and an integer $n$, we define $G^n = \langle x^n, x \in G\rangle$ and $G' = [G, G]$. These are examples of *verbal subgroups* of $G$; see [MKS66, Sect. 2.2] for a full definition and other information on verbal subgroups.

**Lemma 2.7.** *Let $G$ be a group, let $\psi\colon G \to G/K$ be a quotient with kernel $K$, let $V(G)$ be a verbal subgroup of $G$, and let $V(G/K)$ be the analogous verbal subgroup of $G/K$. Then $V(G)$ is a fully invariant subgroup of $G$, $V(G/K) = \psi(V(G))$, and $(G/K)/V(G/K) \cong G/(KV(G))$.*

*Proof.* See [MKS66, p. 74] and [MKS66, ex. 11, p. 80]. □

Finally, we recall the following (easily proven) property of subgroups:

**Lemma 2.8.** *Let $A, B, C$ be subgroups of a group $G$ such that $C \leq A$. Then $(A \cap B)C = A \cap BC$.* □

## 3. Nilpotent and polycyclic groups

In this section, we establish more notation, and assemble some well-known material in the form that we need. We refer the reader to [Rob96] for a more detailed discussion of this material.

**Definition 3.1.** Let $Z(K)$ denote the center of a group $K$. The *upper central series* of $G$ is defined inductively by letting $Z_0(G) = 1$ and letting $Z_{i+1}(G)$ be the preimage of $Z(G/Z_i(G))$ in $G$, and the *lower central series* of $G$ is defined inductively by letting $\gamma_0(G) = G$ and letting $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. The quotients $Z_{i+1}(G)/Z_i(G)$ are the *upper central quotients* of $G$, and the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are the *lower central quotients* of $G$. We say that $G$ is *nilpotent* if $Z_i(G) = G$ for some $i$, or equivalently, if $\gamma_i(G) = 1$ for some $i$ (see [Rob96, 5.1.9]). The *nilpotence class* of $G$ is defined to be the smallest $i$ such that these equalities hold.

Recall that subgroups and quotients of nilpotent groups are nilpotent, and that the torsion elements of a nilpotent group $G$ form a subgroup of $G$ called the *torsion subgroup* of $G$ [Rob96, 5.2.7]. We also recall the following facts.

**Lemma 3.2.** *Let $G$ be a group. Then $Z_j(G/Z_k(G)) = Z_{j+k}(G)/Z_k(G)$.*

*Proof.* See [Rob96, 5.1.11(iv)]. □

**Lemma 3.3.** *Let $G$ be a nilpotent group. The following are equivalent:*

1. *$G$ is torsion-free.*
2. *$Z_1(G)$ is torsion-free.*
3. *All upper central quotients of $G$ are torsion-free.*

*Proof.* Condition 1 implies condition 2 *a fortiori*. Condition 2 implies condition 3 by [Rob96, 5.2.19]. Finally, if condition 3 holds, then any nontrivial $x \in G$ has nontrivial image in some upper central quotient, but no torsion element can have nontrivial image in a torsion-free group. The lemma follows. □

**Corollary 3.4.** *If $G$ is a torsion-free nilpotent group, then $G/Z_1(G)$ is torsion-free.*

*Proof.* By Lemma 3.2, $Z_1(G/Z_1(G)) = Z_2(G)/Z_1(G)$, which is torsion-free by Lemma 3.3. Therefore, $G/Z_1(G)$ is also torsion-free, by another application of Lemma 3.3. □

We will also need the following fact about the verbal subgroup $G^p$ (Definition 2.6) of a nilpotent group.

**Lemma 3.5.** *Let $H$ be a subgroup of a f.g. nilpotent group $G$. Then $H \cap G^p = H^p$ for all sufficiently large primes $p$.*

*Proof.* See [Bau71b, Prop. 2.2]. □

**Definition 3.6.** A group $G$ is *polycyclic* if it has a subnormal series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ whose factors $G_{i+1}/G_i$ are cyclic (that is, a *polycyclic series*).

Recall that subgroups and quotients of polycyclic groups are polycyclic; in fact, a group is polycyclic if and only if it is solvable (has a subnormal series with abelian factors) and all of its subgroups are f.g. (see [Rob96, 5.4.12]). We also recall the following facts.

**Lemma 3.7.** *Let $G$ be a polycyclic group. The number of infinite cyclic factors in a polycyclic series for $G$ is invariant.*

*Proof.* See [Rob96, 5.4.13]. □

**Definition 3.8.** The number of infinite cyclic factors in a polycyclic series for a polycyclic group $G$ is the *Hirsch length* of $G$.

For completeness, we include a proof of the following lemma, also found in [Rob96, Ex. 5.4.10].

**Lemma 3.9.** *Let $H$ be a subgroup of a polycyclic group $G$. Then $H$ and $G$ have the same Hirsch length if and only if $H$ has finite index in $G$.*

*Proof.* Let $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ be a polycyclic series for $G$. Note that if $H_i = H \cap G_i$, then $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ is a polycyclic series for $H$. Applying Lemma 2.3, we see that each $H_i/H_{i-1}$ is isomorphic to a subgroup of $G_i/G_{i-1}$. Therefore, since $H_i/H_{i-1}$ has finite index in $G_i/G_{i-1}$ for all $1 \leq i \leq n$ if and only if the series for $H$ has the same number of infinite cyclic factors as the series for $G$, the lemma follows from the last assertion of Lemma 2.3. □

**Corollary 3.10.** *If $G$ is polycyclic, and $\varphi\colon G \to G$ is a monomorphism, then $\varphi(G)$ has finite index in $G$.*

*Proof.* Since $\varphi(G)$ is isomorphic to $G$, it has the same Hirsch length. □

**Corollary 3.11.** *If $G$ is torsion-free polycyclic, and $\varphi$ is an endomorphism of $G$ such that $\varphi(G)$ has finite index in $G$, then $\varphi$ is a monomorphism.*

*Proof.* If the kernel $K$ of $\varphi$ were nontrivial, and therefore, infinite polycyclic, then by the invariance of the Hirsch length, the Hirsch length of $\varphi(G) \cong G/K$ would be strictly less than the Hirsch length of $G$, which would imply that $\varphi(G)$ has infinite index in $G$. □

**Lemma 3.12.** *Every f.g. nilpotent group has a torsion-free subgroup of finite index.*

*Proof.* This follows *a fortiori* from [Rob96, 5.4.15.(i)]. □

We will also need a result of Malćev that can be stated in the following terms.

**Definition 3.13.** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be group properties (e.g., finiteness, nilpotence, polycyclicity). We say that a group $G$ is $\mathcal{P}_1$*-by-*$\mathcal{P}_2$ if $G$ has a normal subgroup $N$ with property $\mathcal{P}_1$ such that $G/N$ has property $\mathcal{P}_2$.

**Lemma 3.14** (Malćev). *Every polycyclic-by-finite group has a finite index subgroup whose commutator subgroup is nilpotent.*

*Proof.* This can be deduced easily from [Rob96, 15.1.6]. □

## 4. Ascending residual finiteness

**Definition 4.1.** A group $J$ is *ARF* (for Ascending Residually Finite) if for any monomorphism $\varphi\colon J \to J$ and any nontrivial $x \in J$, there exists a finite index normal subgroup $W$ of $J$ such that the following conditions hold:

1. $x \notin W$;
2. $\varphi(W) \subseteq W$;
3. The map $\overline{\varphi}\colon J/W \to J/W$ induced by $\varphi$ is an isomorphism.

If $W$ can always be chosen to be fully invariant, we say that $J$ is *FIARF*. Note that by taking $\varphi$ to be the identity monomorphism, we see that an ARF group must be residually finite.

We will show that polycyclic-by-finite groups are ARF in Section 7. Theorem 1.1 will then follow from:

**Theorem 4.2.** *Let $J$ be a group. Then every ascending HNN extension of $J$ is residually finite if and only if $J$ is ARF.*

*Proof.* Suppose every ascending HNN extension of $J$ is residually finite. Let $\varphi\colon J \to J$ be a monomorphism, and let $x$ be a nontrivial element of $J$. Since the ascending HNN extension $J_\varphi = \langle J, t \,|\, j^t = \varphi(j) \rangle$ is residually finite, there exists a finite quotient $\rho : J_\varphi \to Q$ such that $\rho(x) \neq 1$. Let $W = J \cap \ker \rho$. We claim that $W$ satisifies the conditions of Definition 4.1. First, since $\rho(x) \neq 1$, we have $x \notin W$. Next, if $j \in W$, then since $\rho(j) = 1$, we have

$$\rho(\varphi(j)) = \rho(j^t) = \rho(t)^{-1}\rho(j)\rho(t) = 1, \tag{7}$$

which means that $\varphi(j) \in J \cap \ker \rho = W$. Finally, if $\overline{\varphi}\colon J/W \to J/W$ is the map induced by $\varphi$, we observe that

$$\overline{\varphi}(\rho(j)) = \rho(\varphi(j)) = \rho(t)^{-1}\rho(j)\rho(t). \tag{8}$$

In other words, $\overline{\varphi}$ is conjugation by $\rho(t)$, which is an isomorphism.

Conversely, suppose that $J$ is ARF. Let $\varphi\colon J \to J$ be a monomorphism, and let $a$ be a nontrivial element of the ascending HNN extension $J_\varphi$ (as above). By the normal form theorem, $a = t^r x t^{-s}$ for some $r, s \geq 0$, $x \in J$. On the one hand, if $r - s \neq 0$, then $a$ has nontrivial image in the quotient $\rho : J_\varphi \to \mathbb{Z}_{r+s+1}$ induced by sending all elements of $J$ to 0 and sending $t$ to 1. On the other hand, suppose $r - s = 0$. In that case, after conjugating, we may as well assume that $a = x$. Then, since $J$ is ARF, there is a normal $\varphi$-invariant subgroup $W$ of $J$ such that $x \notin W$, $J/W$ is finite, and $\overline{\varphi}\colon J/W \to J/W$ is an isomorphism. Writing the image of $j \in J$ in $J/W$ as $\overline{j}$, we see that the quotient

$$\overline{J_\varphi} = \left\langle J/W, t \,\middle|\, \overline{j}^{\,t} = \overline{\varphi}(\overline{j}) \right\rangle \tag{9}$$

is finite-by-cyclic (since $\overline{\varphi}$ is an isomorphism), and that $\overline{x} \neq 1$ in $\overline{J_\varphi}$. Therefore, since finite-by-cyclic groups are residually finite, there exists a finite quotient of $\overline{J_\varphi}$ in which $\overline{x}$ has nontrivial image. ☐

## 5. Torsion-free nilpotent groups

In this section, we prove the following theorem.

**Theorem 5.1.** *Every f.g. torsion-free nilpotent group is FIARF.*

Since the subgroup $G^p$ is fully invariant in any group $G$ (Lemma 2.7), it suffices to prove the following theorem.

**Theorem 5.2.** *Let $G$ be a f.g. torsion-free nilpotent group, let $\varphi\colon G \to G$ be a monomorphism, and let $x$ be a nontrivial element of $G$. Then for all sufficiently large primes $p$, the quotient $G/G^p$ is a finite $p$-group, $x \notin G^p$, and the induced map $\varphi\colon G/G^p \to G/G^p$ is an isomorphism.*

For the rest of the section, let $G$ be a f.g. torsion-free nilpotent group, let $\varphi\colon G \to G$ be a monomorphism, and let $Z_i = Z_i(G)$. We begin by showing that $\varphi$ behaves well with respect to upper central quotients.

**Lemma 5.3.** *For all $i$, $\varphi(Z_i) \leq Z_i$ and $\varphi\colon (G/Z_i) \to (G/Z_i)$ is injective.*

*Proof.* Since $G/Z_1$ is also torsion-free nilpotent (Corollary 3.4), by induction, it is enough to show that the lemma holds for $i = 1$. In fact, it is enough to show that

$$\varphi(Z_1) = Z(\varphi(G)) = Z_1 \cap \varphi(G). \tag{10}$$

For then, $\varphi(Z_1)$ is certainly a subgroup of $Z_1$, and if $\varphi(x) \in Z_1$ for some $x \in G$, then $\varphi(x) \in (Z_1 \cap \varphi(G)) = \varphi(Z_1)$ implies that $x \in Z_1$, since $\varphi$ is injective.

To prove (10), we first observe that since $\varphi$ is an embedding, the center of $\varphi(G)$ is precisely the image of the center of $G$, or in other words, $Z(\varphi(G)) = \varphi(Z_1)$. Also, since $\varphi$ is an embedding, and $G$ is torsion-free, we see that $Z(\varphi(G))$ and $Z_1$ are free abelian groups of the same rank. We next observe that $[Z_1 : Z_1 \cap \varphi(G)] \leq [G : \varphi(G)]$, which is finite by Corollary 3.10. Therefore, $Z_1$ and $Z_1 \cap \varphi(G)$ are also free abelian groups of the same rank, which means that $Z_1 \cap \varphi(G)$ has finite index in $Z(\varphi(G))$ as well, since $Z_1 \cap \varphi(G)$ is certainly a subgroup of $Z(\varphi(G))$. Finally, since every element of $Z_1 \cap \varphi(G)$ is central in $\varphi(G)$, we have

$$Z_1 \cap \varphi(G) = Z_1 \cap Z(\varphi(G)). \tag{11}$$

Combining (11) with the second isomorphism theorem, we then have

$$Z(\varphi(G))/(Z_1 \cap \varphi(G)) = Z(\varphi(G))/(Z_1 \cap Z(\varphi(G))) \cong Z(\varphi(G))Z_1/Z_1. \tag{12}$$

It follows that $Z(\varphi(G))Z_1/Z_1$ is a finite subgroup of the torsion-free group $G/Z_1$ (Corollary 3.4), and is therefore trivial. Consequently, $Z(\varphi(G)) = Z_1 \cap \varphi(G)$, and the lemma follows. □

In particular, $\varphi \colon G \to G$ induces a monomorphism $\varphi_i : (Z_i/Z_{i-1}) \to (Z_i/Z_{i-1})$ on each upper central quotient $Z_i/Z_{i-1}$. Moreover, each upper central quotient $Z_i/Z_{i-1}$ is free abelian, so each $\varphi_i$ is a monomorphism of a free abelian group, and we may define

$$d_i = \det(\varphi_i) \tag{13}$$

to be the *upper central determinants* of $\varphi$. Note that each $d_i$ is a nonzero integer.

It will be convenient to have the following definition in the sequel.

**Definition 5.4.** Let $p$ be a prime. We say that $G$ *$p$-reduces* if

$$Z_i \cap G^p = Z_i^p \tag{14}$$

for all $i$.

**Lemma 5.5.** *Let $p$ be a prime. If $G$ $p$-reduces then $G/Z_1$ $p$-reduces.*

*Proof.* Let $\psi \colon G \to G/Z_1$ be the natural map. Lemmas 2.7 and 3.2 imply that $Z_{i-1}(G/Z_1) = \psi(Z_i)$, $(G/Z_1)^p = \psi(G^p)$, and $Z_{i-1}(G/Z_1)^p = \psi(Z_i)^p = \psi(Z_i^p)$. But then, since Lemma 2.8 implies that

$$\psi(Z_i^p) = \psi(Z_i \cap G^p) = (Z_i \cap G^p)Z_1 = Z_i \cap (G^p Z_1) = \psi(Z_i) \cap \psi(G^p), \tag{15}$$

we see that $Z_{i-1}(G/Z_1) \cap (G/Z_1)^p = Z_{i-1}(G/Z_1)^p$. The lemma follows. □

**Lemma 5.6.** *Let $p$ be a prime. If $G$ $p$-reduces then $Z_1G^p/G^p \cong Z_1/Z_1^p$. More generally, if $G$ $p$-reduces then each upper central quotient of $G/G^p$ is the corresponding upper central quotient of $G$ reduced modulo $p$.*

*Proof.* If $G$ $p$-reduces we have

$$Z_1G^p/G^p \cong Z_1/(Z_1 \cap G^p) = Z_1/Z_1^p \tag{16}$$

by the second isomorphism theorem and (14), respectively. The second assertion follows from the first by Lemma 5.5 and an easy induction. □

**Lemma 5.7.** *Suppose that $G$ $p$-reduces where $p$ is a prime that does not divide any of the upper central determinants of $\varphi$. Then the induced map $\varphi\colon (G/G^p) \to (G/G^p)$ is an isomorphism.*

*Proof.* Proceeding by induction on the nilpotence class $n$ of $G$, for $n = 0$, both $G$ and the lemma are trivial, so assume $n > 0$. In that case, $G/Z_1$ is f.g. torsion-free (Corollary 3.4) nilpotent of class $n-1$, $G/Z_1$ $p$-reduces (Lemma 5.5), and $p$ does not divide any of the upper central determinants of $\varphi\colon (G/Z_1) \to (G/Z_1)$, so by induction, the induced map $\varphi\colon (G/Z_1)/(G/Z_1)^p \to (G/Z_1)/(G/Z_1)^p$ is an isomorphism. In fact, since $(G/Z_1)/(G/Z_1)^p \cong G/(Z_1G^p)$ (Lemma 2.7), the induced map $\varphi\colon G/(Z_1G^p) \to G/(Z_1G^p)$ is also an isomorphism.

Next, since $p$ does not divide the determinant of $\varphi_1\colon Z_1 \to Z_1$, the induced map $\varphi_1\colon (Z_1/Z_1^p) \to (Z_1/Z_1^p)$ is an isomorphism. Therefore, since $(Z_1G^p)/G^p \cong Z_1/Z_1^p$ (Lemma 5.6), the induced map $\varphi\colon (Z_1G^p)/G^p \to (Z_1G^p)/G^p$ is an isomorphism. It then follows by Lemma 2.4 that, since the induced maps $\varphi\colon G/(Z_1G^p) \to G/(Z_1G^p)$ and $\varphi\colon (Z_1G^p)/G^p \to (Z_1G^p)/G^p$ are both isomorphisms, the induced map $\varphi\colon G/G^p \to G/G^p$ is also an isomorphism. The lemma follows. □

*Proof of Theorem 5.2.* First, for any prime $p$, $G/G^p$ is a f.g. nilpotent torsion group, and is therefore finite [Rob96, 5.2.18], which means that $G^p$ has finite index in $G$. Next, by applying Lemma 3.5 for each term $Z_i$ of the upper central series, we see that $G$ $p$-reduces for sufficiently large $p$. Furthermore, since each nontrivial $x \in G$ has a nontrivial image in some upper central quotient of $G$, by Lemma 5.6, for sufficiently large $p$, the element $x$ has nontrivial image in one of the upper central quotients of $G/G^p$. Finally, by Lemma 5.7, for sufficiently large $p$, the induced map $\varphi\colon (G/G^p) \to (G/G^p)$ is an isomorphism. The theorem follows. □

## 6. Stabilization

We begin with the following observation, which leads to a definition.

**Theorem 6.1.** *Let $J$ be a group, let $\varphi\colon J \to J$ be a homomorphism, let $V$ be a $\varphi$-invariant normal subgroup of $J$, and let*

$$W = \bigcup_{n=0}^{\infty} \varphi^{-n}(V) = \{x \in J \mid \varphi^n(x) \in V \text{ for some } n\}. \tag{17}$$

*Then $W$ is a $\varphi$-invariant normal subgroup of $J$, and the induced map $\overline{\varphi}\colon J/W \to J/W$ is injective.*

*Proof.* For $x \in J$, if $\varphi^n(x) \in V$ for some $n$, then

$$\varphi^n(\varphi(x)) = \varphi(\varphi^n(x)) \in V, \tag{18}$$

since $V$ is $\varphi$-invariant. Therefore, $W$ is $\varphi$-invariant. Furthermore, $W$ is an increasing union of normal subgroups of $J$, and is therefore itself a normal subgroup of $J$. Finally, suppose that for $x \in J$, $\varphi(x) \in W$. In that case, for some $n$, $\varphi^n(\varphi(x)) = \varphi^{n+1}(x)$ is an element of $V$, which means that $x \in W$. The theorem follows. □

**Definition 6.2.** Let $J$, $\varphi$, $V$, $W$, and $\overline{\varphi}$ be as in Theorem 6.1. In that case, we say that $W$ is the *stable kernel of* $\varphi$ *relative to* $V$, and we say that $\overline{\varphi}$ is the *stabilization* of $\varphi$ relative to $V$.

The idea of stabilization is the key to the following theorem.

**Theorem 6.3.** *Let $J$ be a group, and let $G$ be a fully invariant subgroup of $J$. Suppose that:*

1. *$G$ is FIARF; and*
2. *Every quotient of a finite-by-$(J/G)$ group is ARF.*

*Then $J$ is ARF.*

*Proof.* Let $\varphi\colon J \to J$ be a monomorphism, let $K$ be the stable kernel of $\varphi$ relative to $G$, and let $x$ be a nontrivial element of $J$. We then have two cases:

1. If $x \in K$, then $\varphi^n(x) \in G$ for some $n$, and since $\varphi$ is a monomorphism, $\varphi^n(x) \neq 1$. Then, by hypothesis 1, there is a finite index fully invariant subgroup $V_0$ of $G$ such that $\varphi^n(x) \notin V_0$ and $\varphi\colon G/V_0 \to G/V_0$ is an isomorphism induced by the restriction of $\varphi$ to $G$. Let $V$ be be the stable kernel of $\varphi$ relative to $V_0$. Note that since $\varphi\colon G/V_0 \to G/V_0$ is injective, $V \cap G = V_0$. It follows that $\varphi^n(x) \notin V$, since $\varphi^n(x) \notin V_0$ and $\varphi^n(x) \in G$. Therefore, since the induced endomorphism $\overline{\varphi}\colon J/V \to J/V$ is injective (Theorem 6.1), $x \notin V$.
2. If $x \notin K$, let $V_0 = G$, and let $V = K$.

In either case, it now follows that $J/V$ is a quotient of the finite-by-$(J/G)$ group $J/V_0$, that the induced endomorphism $\varphi\colon J/V \to J/V$ is injective (Theorem 6.1), and that $x \neq 1$ in $J/V$. Therefore, by hypothesis 2, there exists a finite index $\varphi$-invariant subgroup $U \lhd J/V$ such that $x \notin U$ and $\varphi\colon (J/V)/U \to (J/V)/U$ is an isomorphism. If we then let $W$ be the preimage of $U$ in $J$, we see that $W$ is normal of finite index in $J$, $x \notin W$, $W$ is $\varphi$-invariant, and the induced map $\varphi\colon J/W \to J/W$ is an isomorphism. The theorem follows. ☐

## 7. Proof of Theorem 1.1

Having previously established the torsion-free nilpotent case, we now use Theorem 6.3 to expand the class of groups we know to be ARF until we obtain Theorem 1.1. We start with a trivial case.

**Lemma 7.1.** *Every finite group is ARF.*

*Proof.* Every monomorphism from a finite group to itself is an isomorphism. ☐

**Lemma 7.2.** *Every f.g. nilpotent-by-finite group is ARF.*

*Proof.* Let $J$ be a f.g. group with a nilpotent normal subgroup $N$ such that $J/N$ is finite. By Lemma 3.12, there exists a torsion-free subgroup $N_0$ of finite index in $N$, which means that, by Lemma 2.2, there exists a fully invariant torsion-free nilpotent subgroup $N_1$ of finite index in $J$. Therefore, since $N_1$ is FIARF (Theorem 5.1), and since quotients of finite-by-$J/N_1$ groups are finite, and therefore ARF (Lemma 7.1), Theorem 6.3 implies that $J$ is ARF. ☐

**Lemma 7.3.** *Every f.g. (finite-by-abelian)-by-finite group is ARF.*

*Proof.* By Lemma 7.2, it is enough to show that every f.g. finite-by-abelian group has a nilpotent subgroup of finite index. Let $J$ be a f.g. group with a finite normal subgroup $K$ such that $J/K$ is abelian, and let $C = C_J(K)$ be the centralizer of $K$ in $J$. Since $K$ is normal and finite, each of the finitely many elements of $K$ has only finitely many conjugates in $J$, which means that $C$ has finite index in $J$. Finally, we observe that $C/(C \cap K) \cong CK/K$ is abelian, and that $C \cap K$ is central in $C$, which means that $C$ is nilpotent of class $\leq 2$. □

*Proof of Theorem 1.1.* Let $J$ be a polycyclic-by-finite group. By Lemma 3.14, there exists a finite index subgroup $H$ of $J$ such that $H'$ is nilpotent. In fact, by Lemma 2.2, we may assume that $H$ is fully invariant in $J$, which means that $H'$ is also fully invariant in $J$ (Lemma 2.7). Furthermore, since $H'$ is itself polycyclic-by-finite, and therefore f.g., by Lemma 3.12, $H'$ has a torsion-free subgroup $G$ of finite index. By Lemma 2.2, we may also assume that $G$ is fully invariant in $H'$, and therefore, in $J$. Putting this all together, since $H'/G$ is finite, $H/H'$ is abelian, $J/H$ is finite, and all subgroups are fully invariant in $J$, we see that any quotient of a finite-by-$J/G$ group is f.g. (finite-by-abelian)-by-finite. Furthermore, by Theorem 5.1, $G$ is FIARF, and by Lemma 7.3, any quotient of a finite-by-$J/G$ group is ARF. Therefore, by Theorem 6.3, $J$ is ARF. □

## 8. Ascending HNN extensions of other groups

In [GMSW01], it was conjectured that:

**Conjecture 8.1.** *Every ascending HNN extension $F_\varphi$ of a f.g. free group $F$ is residually finite.*

In this section, we use Theorem 1.1 to make some partial progress towards Conjecture 8.1 and related problems. More specifically, let $G_{\text{ab}}$ denote the abelianization $G/G'$ of a group $G$. Our main result in the direction of Conjecture 8.1 is:

**Theorem 8.2.** *Let $G$ be a f.g. residually torsion-free nilpotent group, let $\varphi\colon G \to G$ be a monomorphism, and suppose that $[G_{\text{ab}} : \varphi_{\text{ab}}(G_{\text{ab}})]$ is finite. Then $G_\varphi$ is residually finite.*

Most notably, since free groups are residually free nilpotent (see [MKS66, Sect. 5.5]), Theorem 8.2 immediately implies the following unpublished result of Mark Sapir [Sap].

**Theorem 8.3** (Sapir)**.** *If $F$ is a f.g. free group, and $\varphi\colon F \to F$ is a monomorphism such that the induced map $\varphi_{\text{ab}}\colon F_{\text{ab}} \to F_{\text{ab}}$ is also a monomorphism, then $F_\varphi$ is residually finite.* □

Note that Theorem 8.2 can also be applied in many other situations. For example, a *right-angled Artin group* is a group with a presentation of the form

$$\langle a_1, \ldots, a_n \mid [a_i, a_j], (i,j) \in S\rangle, \tag{19}$$

where $S$ is a subset of $\{(i,j) \mid 1 \leq i < j \leq n\}$. By [Gre90], every right-angled Artin group is residually nilpotent. Furthermore, there are many monomorphisms of right-angled Artin groups that satisfy the criterion of Theorem 8.2; for instance, take the monomorphism induced by $a_i \mapsto a_i^{m_i}$, where each $m_i$ is a nonzero integer. One can verify that this is a monomorphism by applying the normal form theorem for right-angled Artin groups [Gre90, HW99].

In the rest of this section, we prove Theorem 8.2. Throughout, we abbreviate $\gamma_i(G)$ as $\gamma_i$. We begin with the following lemma.

**Lemma 8.4.** *Let $G$ be f.g. nilpotent, let $\alpha\colon G \to G/G'$ be the abelianization of $G$, and let $H$ be a subgroup of $G$ such that $\alpha(H)$ has finite index in $\alpha(G)$. Then $H$ has finite index in $G$.*

*Proof.* We first observe that every conjugate of $H$ has image $\alpha(H)$ in $\alpha(G)$. Therefore, by replacing $H$ with the intersection of all of its conjugates, we may assume without loss of generality that $H$ is normal in $G$.

For $i \geq 0$, define $H_i$ recursively by $H_0 = H$ and $H_{i+1} = [G, H_i]$. We then see that:

1. Using the identity $[x,y]^g = [x^g, y^g]$, an easy induction, starting with $H_0 = H \lhd G$, shows that $H_i \lhd G$ for each $i$.
2. Since $H_i \lhd G$, we have $[g,h] \in H_i$ for all $g \in G$ and $h \in H_i$. Another easy induction therefore shows that $H_i \leq H$ for each $i$.
3. Finally, note that $H_0 = H \cap \gamma_0$, and that if $H_i \leq H \cap \gamma_i$, then

$$H_{i+1} = [G, H_i] \leq [G, H \cap \gamma_i] \leq [G, \gamma_i] = \gamma_{i+1} \tag{20}$$

implies that $H_{i+1} \leq H \cap \gamma_{i+1}$, since $H_{i+1} \leq H$. By induction, we see that $H_i \leq H \cap \gamma_i$ for all $i \geq 0$.

Now, by Lemma 2.3, to show that $[G : H]$ is finite, it is enough to show that

$$(\gamma_i/\gamma_{i+1})/((H \cap \gamma_i)/\gamma_{i+1}) \cong \gamma_i/(H \cap \gamma_i)\gamma_{i+1} \tag{21}$$

is finite for all $i \geq 0$. Furthermore, since $H_i \leq H \cap \gamma_i$, *a fortiori*, it is enough to show that $\gamma_i/H_i\gamma_{i+1}$ is finite for all $i \geq 0$. In fact, since $\gamma_0/H_0\gamma_1 = G/HG'$ is finite by hypothesis, and each $\gamma_i/H_i\gamma_{i+1}$ is f.g. abelian, by induction, it is enough to show that if $\gamma_i/H_i\gamma_{i+1}$ has finite exponent, then $\gamma_{i+1}/H_{i+1}\gamma_{i+2}$ has finite exponent.

So suppose that $\gamma_i/H_i\gamma_{i+1}$ has finite exponent $n$. Since $[G, \gamma_{i+1}] = \gamma_{i+2}$, we see that $\gamma_{i+1}$ is central in $G$, modulo $\gamma_{i+2}$. Therefore, using the identity $[x, yz] = [x,z][x,y]^z$, we see that

$$[x, yz] = [x,y][x,z] \pmod{\gamma_{i+2}} \tag{22}$$

for all $x \in G$ and $y, z \in \gamma_i$.

Consider any generator $[g, g_i]$ of $\gamma_{i+1}/H_{i+1}\gamma_{i+2}$, where $g \in G$ and $g_i \in \gamma_i$. Repeatedly applying (22), we see that

$$[g, g_i]^n = [g, g_i^n] \pmod{\gamma_{i+2}}. \tag{23}$$

However, since $n$ is the exponent of $\gamma_i/H_i\gamma_{i+1}$, we have $g_i^n = h_i g_{i+1}$ for some $h_i \in H_i$ and $g_{i+1} \in \gamma_{i+1}$. Therefore,

$$[g, g_i]^n = [g, h_i g_{i+1}] = [g, h_i][g, g_{i+1}] \pmod{\gamma_{i+2}}, \tag{24}$$

which is an element of $H_{i+1}\gamma_{i+2}$. We conclude that each generator of $\gamma_{i+1}/H_{i+1}\gamma_{i+2}$ has order dividing $n$, which means that the f.g. abelian group $\gamma_{i+1}/H_{i+1}\gamma_{i+2}$ has exponent dividing $n$. □

*Proof of Theorem 8.2.* Let $x$ be a nontrivial element of $G$, and choose $n$ such that $x \notin \gamma_n$. Let $Q = G/\gamma_n$, and consider the projection of $\varphi\colon G \to G$ to $\overline{\varphi}\colon Q \to Q$. Since the abelianization of $\varphi$, and therefore, $\overline{\varphi}$, has non-zero determinant, Lemma 8.4 implies that $\overline{\varphi}(Q)$ has finite index in $Q$. Since $Q$ is torsion-free polycyclic, Corollary 3.11 implies that $\overline{\varphi}$ is a monomorphism. Therefore, we see that $Q_{\overline{\varphi}}$ is an ascending HNN extension, and that $x$ survives in the quotient $G_\varphi \to Q_{\overline{\varphi}}$. However, by Theorem 1.1, $Q_{\overline{\varphi}}$ is residually finite, so $x$ survives in some finite quotient of $G_\varphi$. □

**Remark 8.5.** In [Bau72], G. Baumslag gave an example of an infinitely generated free-by-cyclic group with no non-cyclic finite quotient. Therefore, Conjecture 8.1 does not hold if $F$ is not f.g., even if $\varphi$ is an isomorphism. However, in contrast, Baumslag has also shown that a f.g. free-by-cyclic group is residually finite [Bau71a]. It therefore seems likely that if residual finiteness holds for the class of ascending HNN extensions of f.g. free groups, then residual finiteness will also hold for the somewhat more general class of f.g. ascending HNN extensions of free groups.

**Remark 8.6.** Another family of properly ascending HNN extensions was shown to be residually finite in [Wis99, Example 2.3]. The approach there is of an entirely different nature, and includes examples where $\varphi(F) \subset F'$, making $\varphi_{\mathrm{ab}}$ singular.

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