PERMUTATION TECHNIQUES FOR COSET REPRESENTATIONS OF MODULAR SUBGROUPS

TIM HSU

1. Introduction

We recall that dessins d'enfants may be considered as one of the following equivalent classes of objects. Let G be an appropriate "cartographical" group. (We may take $G = \Gamma(2)$ for general dessins, $G = \Gamma_0(2)$ for pre-clean dessins, and $G = \mathbf{PSL}_2(\mathbf{Z})$ for pre-clean dessins whose valencies at the "other" type of vertex divide 3.)

1. Coverings of the Riemann sphere, branched (in a manner consistent with G) only above $\{0, 1, \infty\}$, along with a choice of (unramified) basepoint.
2. Drawings (structured in a manner consistent with G) on connected oriented surfaces.
3. Subgroups of G.
4. Transitive permutation representations of G, along with a choice of basepoint.

The correspondences among these classes of objects have been explained many times, so we refer the reader to Birch [2], Jones and Singerman [5], and Schneps [7] for details, mentioning only one piece of terminology: In describing the correspondence between (3) and (4), we call the basepointed transitive permutation representation of G corresponding to a subgroup $G_1 \leq G$ the *coset representation of* G_1 *as a subgroup of* G.

This article describes some computational methods for working with dessins and similar structures. Briefly, one might say that the questions examined here arise from subgroups (viewpoint (3)), and the answers come from looking at their coset representations (viewpoint (4)). We hope that the methods presented here can be used to address other questions about dessins by looking at them in terms of their coset representations.

Throughout, we will describe coset representations of subgroups of $G = \mathbf{PSL}_2(\mathbf{Z})$ using the following notation. Let

$$\gamma_0 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

γ_0, γ_1, and γ_∞ satisfy the relations

$$1 = \gamma_0^3 = \gamma_1^2 = \gamma_0\gamma_1\gamma_\infty \tag{1.2}$$

in G, and in fact, given the generators γ_0, γ_1, and γ_∞, (1.2) is a set of defining relations for G. Therefore, to specify a basepointed transitive permutation representation of G, it is enough to specify the images σ_0, σ_1, and σ_∞ of γ_0, γ_1, and

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γ_∞. In other words, to obtain a basepointed transitive permutation representation of G, we need only choose transitive permutations σ_0, σ_1, and σ_∞ which respect (1.2). (By convention, we take the point numbered 1 as our basepoint.)

For instance,

$$\sigma_0 = (1\ 2\ 3), \qquad \sigma_1 = (1\ 2), \qquad \sigma_\infty = (2\ 3), \tag{1.3}$$

is the coset representation of $\Gamma_0(2)$ as a subgroup of G.

By convention, we draw the dessin corresponding to the coset representation of a subgroup of G by saying that σ_0 describes how edges are arranged counterclockwise around vertices marked with a $\circ$, σ_1 describes how edges are arranged counterclockwise around vertices marked with a $|$, and σ_∞ how edges are arranged counterclockwise around a 2-cell. For example, the dessin corresponding to (1.3) is shown in Figure 1.

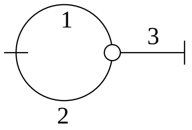

FIGURE 1. Dessin for $\Gamma_0(2)$

2. IDENTIFYING CONGRUENCE SUBGROUPS

Let $G = \mathbf{PSL}_2(\mathbf{Z})$. In this section, we describe an algorithmic answer to the following problem.

- Given the coset representation of a subgroup $G_1 \leq G$ of finite index, determine if G_1 is a congruence subgroup of G. That is, determine if G_1 contains $\Gamma(N)$ for some value of N.

We begin with the result often known as Wohlfahrt's Theorem [13], for which we need the following definition.

Definition 2.1. Let G_1 be a modular subgroup of finite index, and consider the coset representation of G_1 in G, using the notation of Section 1. The (generalized) level of G_1 is defined in one of the following equivalent ways, roughly corresponding to the viewpoints listed in Section 1.

1. The least common multiple of the cusp widths of the quotient of the upper half-plane by G_1.
2. The least common multiple of the numbers of sides of the 2-cells of the dessin corresponding to G_1. (Note that number of sides of a 2-cell is half the number of edges which touch it, counting multiplicities.)
3. The smallest value of N such that all conjugates of γ_∞^N are contained in G_1.
4. The order of the permutation σ_∞.

Theorem 2.2 (Wohlfahrt's Theorem). *Let G_1 be a modular subgroup of generalized level N. Then G_1 is a congruence subgroup if and only if G_1 contains $\Gamma(N)$.* □

The following can then be shown. (See [4] for a proof, in slightly different notation.)

Theorem 2.3. *Let G_1 be a modular subgroup of level N, and let*

$$\langle \sigma_0, \sigma_1, \sigma_\infty \,|r_1(\sigma_0, \sigma_1, \sigma_\infty),\ r_2(\sigma_0, \sigma_1, \sigma_\infty),\ \dots\rangle \tag{2.1}$$

be a presentation for $\mathbf{SL}_2(\mathbf{Z}/N)/\{\pm I\}$ which is fulfilled by taking $\sigma_0 = \gamma_0 \pmod N$, $\sigma_1 = \gamma_1 \pmod N$, and $\sigma_\infty = \gamma_\infty \pmod N$. Then G_1 is a congruence subgroup if and only if the coset representation of G_1 respects the relations $\{r_i\}$. ☐

Implementation. To apply Theorem 2.3 effectively, we need a uniform method of presenting $\mathbf{SL}_2(\mathbf{Z}/N)/\{\pm I\}$. See [4] for one such method. Note that with the uniform method given there, one can write a program to determine if a modular subgroup is congruence by checking no more than 12 relations.

Example 2.4. We illustrate Theorem 2.3 by examining two subgroups of index 18 and generalized level 6, one of which is congruence, and the other of which is not. Let G_1 be the modular subgroup with coset representation

$$\begin{aligned} \sigma_0 &= (1\ 13\ 8)(2\ 7\ 16)(3\ 15\ 10)(4\ 9\ 18)(5\ 17\ 12)(6\ 11\ 14), \\ \sigma_1 &= (1\ 7)(2\ 15)(3\ 9)(4\ 17)(5\ 11)(6\ 13)(8\ 18)(10\ 14)(12\ 16), \\ \sigma_\infty &= (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)(13\ 14\ 15\ 16\ 17\ 18), \end{aligned} \tag{2.2}$$

and let G_2 be the modular subgroup with coset representation

$$\begin{aligned} \sigma_0 &= (1\ 15\ 17)(2\ 16\ 6)(3\ 5\ 7)(4\ 12\ 8)(9\ 11\ 13)(10\ 18\ 14), \\ \sigma_1 &= (1\ 16)(2\ 5)(3\ 12)(4\ 7)(6\ 15)(8\ 11)(9\ 18)(10\ 13)(14\ 17), \\ \sigma_\infty &= (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)(13\ 14\ 15\ 16\ 17\ 18). \end{aligned} \tag{2.3}$$

The dessins for (2.2) and (2.3) are shown in Figures 2 and 3. Since both dessins have genus 1, we have drawn them as polygons with identified boundaries; identifications are indicated by edge numbers which appear twice.

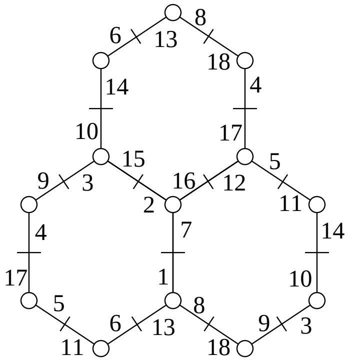

FIGURE 2. Dessin for G_1

It can be shown by coset enumeration that $\mathbf{SL}_2(\mathbf{Z}/6)/\{\pm I\}$ has a presentation with generators σ_0, σ_1, and σ_∞, and defining relations

$$1 = \sigma_0^3 = \sigma_1^2 = \sigma_0\sigma_1\sigma_\infty = \sigma_\infty^6, \tag{2.4}$$

$$1 = [\sigma_\infty^2, \sigma_1^{-1}\sigma_\infty^{-3}\sigma_1], \tag{2.5}$$

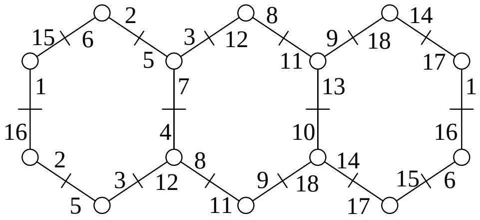

FIGURE 3. Dessin for G_2

where $[x,y] = x^{-1}y^{-1}xy$. However, the coset representation of a subgroup of generalized level 6 satisfies (2.4) automatically. Therefore, for groups of level 6, Theorem 2.3 boils down to testing (2.5). For G_1, we have

$$[\sigma_\infty^2, \sigma_1^{-1}\sigma_\infty^{-3}\sigma_1] = (), \tag{2.6}$$

and for G_2, we have

$$[\sigma_\infty^2, \sigma_1^{-1}\sigma_\infty^{-3}\sigma_1] = (1\ 13\ 7)(2\ 8\ 14)(3\ 15\ 9)(4\ 10\ 16)(5\ 17\ 11)(6\ 12\ 18), \tag{2.7}$$

so G_1 is congruence and G_2 is noncongruence.

3. ENLARGING SUBGROUPS

As a natural extension to the results in the previous section, in this section, we solve the following family of problems. Let $G = \mathbf{PSL}_2(\mathbf{Z})$.

1. Given an arbitrary subgroup G_1 of finite index in G, find the coset representation of $\langle G_1, \Gamma(N)\rangle$ (the closure of the two groups in G).
2. Given an arbitrary subgroup G_1 of finite index in G, find the coset representation of the smallest congruence subgroup $H \leq G$ containing G_1. H is known as the *congruence closure* of G_1. Note that Wohlfahrt's Theorem reduces this problem to problem (1).
3. Given an arbitrary subgroup G_1 of finite index in G, find the coset representation of the smallest subgroup of G which contains G_1 and has generalized level dividing N. In other words, if $U(N)$ is the normal closure of γ_∞^N in G, find the coset representation of $\langle G_1, U(N)\rangle$.

We can generalize these three problems as:

4. Given an arbitrary subgroup G_1 of finite index in G and a "known" subgroup Γ of G, find the coset representation of $\langle G_1, \Gamma\rangle$.

Note that this new formulation of the problem is no longer restricted to the case where $G = \mathbf{PSL}_2(\mathbf{Z})$, and in fact becomes quite a general problem on dessins and related structures.

The goal of this section is to show how we may employ a slight variation of an algorithm which is well-known in another context to solve the above problems efficiently. In doing so, we will also show how to solve the following problem.

5. Given an arbitrary dessin D, find the largest dessin which is covered by D in a manner which identifies two chosen flags of D.

For the reader who may be less familiar with permutation groups, we present a brief summary (after Wielandt [11, §7]) of the theory of imprimitivity, using the language of *lattices*. (For a standard reference, see Crawley and Dilworth [3].) Meet and join in a lattice are denoted by $\wedge$ and $\vee$, respectively. The partial order in a lattice is denoted by $\leq$. Let Ω be a set. A *partition* of Ω is a set P of nonempty disjoint subsets of Ω (the *members* of P) whose union is Ω. We use $\mathrm{Part}(\Omega)$ to denote the set of partitions of Ω. $\mathrm{Part}(\Omega)$ has a natural lattice structure (Crawley and Dilworth [3, Ch. 12]) which we describe with the following examples.

Example 3.1. Let $\Omega = \{1,2,3,4,5,6\}$, let $E = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$, let $P = \{\{1,2\},\{3\},\{4\},\{5,6\}\}$, let $Q = \{\{1\},\{2\},\{3,4\},\{5\},\{6\}\}$, and let $B = \{\{1,2\},\{3,4\},\{5,6\}\}$. Then $E, P, Q, B \in \mathrm{Part}(\Omega)$, $E \leq P \leq B \leq \{\Omega\}$, $E \leq Q \leq B$, $P \vee Q = B$, and $P \wedge Q = E$.

Convention 3.2. When Ω is understood, we write partitions of Ω using the conventions for writing cycle shapes of permutations. That is, we omit one-element subsets, commas between subsets, and the outermost pair of brackets, denoting the partition of Ω into one-element subsets by empty brackets. For instance, in the above example, we write $E = \{\}$ and $P = \{1,2\}\{5,6\}$.

We also recall that $\mathrm{Part}(\Omega)$ is a *complete* lattice (Crawley and Dilworth [3, Ch. 12]). In particular, in every sublattice L of $\mathrm{Part}(\Omega)$, the meet of all elements of L is the unique minimal element of L.

Now suppose a group G acts transitively on Ω. We say that a partition B of Ω is a *system of imprimitivity for G on Ω* if the action of G permutes the members of B. We use $\mathrm{Imp}_G(\Omega)$ to denote the set of systems of imprimitivity for G on Ω. Note that $\mathrm{Imp}_G(\Omega)$ is a sublattice of $\mathrm{Part}(\Omega)$.

Example 3.3. Let Ω and B be as in Example 3.1, and let G be the group generated by the permutations $(1\ 3\ 5)(2\ 4\ 6)$ and $(1\ 4)(2\ 3)(5\ 6)$. Then $B \in \mathrm{Imp}_G(\Omega)$.

Finally, let 1 be an element of Ω, and let $G_1 = \mathrm{Stab}_G(1)$ (the stabilizer of 1 in G).

Definition 3.4. We define a map Φ_1 from $\mathrm{Imp}_G(\Omega)$ to subgroups of G by saying that, for $B \in \mathrm{Imp}_G(\Omega)$, $\Phi_1(B)$ is the stabilizer of the member of B containing 1.

Theorem 3.5. *Φ_1 is an isomorphism (of lattices) between $\mathrm{Imp}_G(\Omega)$ and the lattice of subgroups of G containing G_1.*

Proof. See Wielandt [11, Thm. 7.5] or Aschbacher [1, (5.18)]. □

For the rest of this section, let G be a group acting transitively on a set Ω, let 1 be an element of Ω, let $G_1 = \mathrm{Stab}_G(1)$, let Γ be an arbitrary subgroup of G, and let Γ^G denote the normal closure of Γ in G. For $H \leq G$ and $x \in \Omega$, we use $O_H(x)$ to denote the partition which has $\mathrm{Orb}_H(x)$ (the orbit of x under the action of H) as one of its members, and the single-element subsets of Ω not contained in $\mathrm{Orb}_H(x)$ as its other members. We use O_H to denote the partition of Ω into its H-orbits.

We are now ready to solve problems (1)–(4) in terms of the theory of imprimitivity. The key is the following lemma.

Lemma 3.6. *The isomorphism Φ_1 (Definition 3.4) sends the lattice of all $B \in \mathrm{Imp}_G(\Omega)$ such that $O_\Gamma(1) \leq B$ onto the lattice of all subgroups of G containing both G_1 and Γ.*

Proof. Suppose $B \in \mathrm{Imp}_G(\Omega)$, and b is the member of B containing 1. If $O_\Gamma(1) \leq B$, then every element of Γ fixes b, that is, $\Gamma \leq \mathrm{Stab}_G(b)$. Conversely, if $\Gamma \leq \mathrm{Stab}_G(b)$, then $\mathrm{Orb}_\Gamma(1)$ must be contained in b, that is, $O_\Gamma(1) \leq B$. □

Theorem 3.7. *Let* B_O *be the smallest* $B \in \mathrm{Imp}_G(\Omega)$ *such that* $O_\Gamma(1) \leq B$, *and let* b_O *be the member of* B_O *containing 1. Then* $\mathrm{Stab}_G(b_O) = \langle G_1, \Gamma \rangle$.

Proof. The theorem follows from Lemma 3.6 because the minimal elements of isomorphic lattices must correspond. ☐

To prove the analogous theorem for Γ^G, we need a lemma about basepoints.

Lemma 3.8. *Suppose* $B \in \mathrm{Imp}_G(\Omega)$. *Then for any* $g \in G$, $O_{g\Gamma g^{-1}}(1) \leq B$ *if and only if* $O_\Gamma(1g) \leq B$.

Proof. For $x \in \Omega$, let b_x be the member of B containing x. We first note that since $B \in \mathrm{Imp}_G(\Omega)$, $b_1 g = b_{1g}$ for all $g \in G$. Now, $\mathrm{Orb}_{g\Gamma g^{-1}}(1)$ is contained in b_1 if and only if $1g\gamma g^{-1} \in b_1$ for all $\gamma \in \Gamma$. However, the latter is true if and only if $(1g)\gamma \in (b_1 g)$ for all $\gamma \in \Gamma$, that is, if and only if $\mathrm{Orb}_\Gamma(1g)$ is contained in b_{1g}. ☐

Theorem 3.9. *Let* B_O *be the smallest* $B \in \mathrm{Imp}_G(\Omega)$ *such that* $O_\Gamma \leq B$, *and let* b_O *be the member of* B_O *containing 1. Then* $\mathrm{Stab}_G(b_O) = \langle G_1, \Gamma^G \rangle$.

Proof. Lemma 3.6 implies that Φ_1 maps the lattice of all subgroups of G containing $\langle G_1, \Gamma^G \rangle$ to the lattice of all $B \in \mathrm{Imp}_G(\Omega)$ such that $O_{g\Gamma g^{-1}}(1) \leq B$ for all $g \in G$. However, using Lemma 3.8 and the fact that G is transitive,

$$\begin{aligned} &\{B \in \mathrm{Imp}_G(\Omega) \mid O_{g\Gamma g^{-1}}(1) \leq B, \ \forall g \in G\} \\ =&\{B \in \mathrm{Imp}_G(\Omega) \mid O_\Gamma(1g) \leq B, \ \forall g \in G\} \\ =&\{B \in \mathrm{Imp}_G(\Omega) \mid O_\Gamma \leq B\} \end{aligned}$$

The theorem then follows from the correspondence of minimal elements. ☐

When $G = \mathbf{PSL}_2(\mathbf{Z})$, it turns out that Theorem 3.9 is particularly useful for solving the motivating problems of Section 1. For problem (3), we may apply the theorem with $\Gamma = \langle \gamma_\infty^N \rangle$. For problem (1), we may let S be the set of relators of a presentation for $\mathbf{SL}_2(\mathbf{Z}/N)/\{\pm I\}$, and apply the theorem with Γ equal to the group generated by the elements of S.

For comparison, it is worth noting that the congruence test of Section 2 is a trivial case of Theorem 3.9. To be precise, let S be a set of relators of a presentation for $\mathbf{SL}_2(\mathbf{Z}/N)/\{\pm I\}$ (i.e., a set of normal generators of $\Gamma(N)$ in $\mathbf{PSL}_2(\mathbf{Z})$), and let $\Gamma = \langle S \rangle$. Then if G_1 has generalized level N, G_1 is congruence if and only if O_Γ is the partition of Ω into its one-element subsets. More generally, the congruence subgroups of $\mathbf{PSL}_2(\mathbf{Z})$ which contain G_1 correspond (under the map Φ_1) precisely with those $B \in \mathrm{Imp}_G(\Omega)$ such that $O_\Gamma \leq B$.

Implementation. Examination of Theorems 3.7 and 3.9 shows that we have reduced our original problems to the following problem.

- Given a finitely generated group G which acts transitively on a finite set Ω, and given a partition O of Ω, determine the smallest system of imprimitivity $\geq O$. (In Theorem 3.7, $O = O_\Gamma(1)$, and in Theorem 3.9, $O = O_\Gamma$.)

Consideration of Theorem 3.7 also shows that solving this problem is exactly what we need to solve problem (5) above; take O to be the partition $\{a, b\}$, where a and b are the flags which are to be identified.

It turns out that the above problem has been studied extensively in computational group theory. For instance, the `BlocksSeed` function for permutation groups in GAP [9] is a readily available algorithm for solving the above problem, and Sims' COINCIDENCE algorithm [10, Ch. 4] is an effectively linear-time algorithm for doing the same thing. Note that both algorithms must be modified slightly to take an arbitrary partition as their input. (An appropriately modified version of `BlocksSeed` is available from the author.)

We demonstrate the use of our results with the following examples.

Example 3.10. Let $G = \mathbf{PSL}_2(\mathbf{Z})$, let $\Omega = \{1, \dots, 18\}$, and let $G_3 \leq G$ be the subgroup of index 18 defined by

$$\begin{aligned}
\sigma_0 &= (1\ 11\ 6)(2\ 5\ 12)(3\ 15\ 8)(4\ 10\ 16)(7\ 18\ 13)(9\ 14\ 17),\\
\sigma_1 &= (1\ 5)(2\ 11)(3\ 10)(4\ 15)(6\ 18)(7\ 12)(8\ 14)(9\ 16)(13\ 17),\\
\sigma_\infty &= (1\ 2)(3\ 4)(5\ 6\ 7)(8\ 9\ 10)(11\ 12\ 13\ 14\ 15\ 16\ 17\ 18).
\end{aligned}$$

The dessin for G_3 is shown in Figure 4.

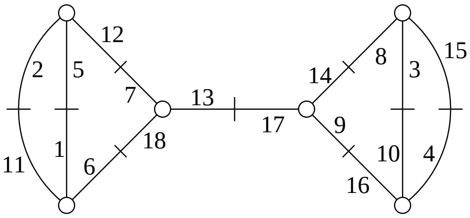

FIGURE 4. Dessin for G_3

To determine the coset representation of the group $H_1 = \langle G_3, \gamma_\infty \rangle$, we apply Theorem 3.7 with $\Gamma = \langle \gamma_\infty \rangle$, which means that $O = O_\Gamma(1) = \{1, 2\}$ (using Convention 3.2). A computer calculation shows that the smallest system of imprimitivity $\geq O$ is $\{\Omega\}$, which means that $H_1 = G$.

On the other hand, to determine the coset representation of the group $H_2 = \langle G_3, \gamma_1^{-1}\gamma_\infty\gamma_1 \rangle$, we apply Theorem 3.7 with $\Gamma = \langle \gamma_1^{-1}\gamma_\infty\gamma_1 \rangle$, which means that $O = O_\Gamma(1) = \{1, 12, 18\}$. A computer calculation shows that the smallest system of imprimitivity $\geq O$ is

$$\{\{1, 3, 12, 14, 16, 18\}, \{2, 4, 11, 13, 15, 17\}, \{5, 6, 7, 8, 9, 10\}\}.$$

Let $a = \{1, 3, 12, 14, 16, 18\}$, $b = \{2, 4, 11, 13, 15, 17\}$, and $c = \{5, 6, 7, 8, 9, 10\}$. We see that σ_0 induces the permutation $(a\ b\ c)$, and σ_1 induces the permutation $(a\ c)$. It is not too hard to see that this implies $H_2 = \gamma_1\Gamma_0(2)\gamma_1^{-1}$. (Compare (1.3).)

Example 3.11. Let $G = \mathbf{PSL}_2(\mathbf{Z})$, let $\Omega = \{1, \dots, 18\}$, and let $G_2 \leq G$ be the subgroup defined by (2.3). Let H be the congruence closure of G_2. Since (2.4) and (2.5) form a set of defining relations for $\mathbf{SL}_2(\mathbf{Z}/6)/\{\pm I\}$, and (2.4) is satisfied by the coset representation of any group of level 6, to find H, we apply Theorem 3.9 with $\Gamma = \langle [\gamma_\infty^2, \gamma_1^{-1}\gamma_\infty^{-3}\gamma_1] \rangle$. However, since

$$[\sigma_\infty^2, \sigma_1^{-1}\sigma_\infty^{-3}\sigma_1] = (1\ 13\ 7)(2\ 8\ 14)(3\ 15\ 9)(4\ 10\ 16)(5\ 17\ 11)(6\ 12\ 18), \tag{3.1}$$

we just have to find the smallest system of imprimitivity $\geq O$, where

$$O = O_\Gamma = \{\{1, 7, 13\}\{2, 8, 14\}\{3, 9, 15\}\{4, 10, 16\}\{5, 11, 17\}\{6, 12, 18\}\}. \tag{3.2}$$

However, O is actually invariant under the action of G, so to find the coset representation of H, we just have to look at how G acts on the members of O. Let $a = \{1, 7, 13\}$, $b = \{2, 8, 14\}$, $c = \{3, 9, 15\}$, $d = \{4, 10, 16\}$, $e = \{5, 11, 17\}$, and

$f = \{6, 12, 18\}$. Then σ_0 induces $(a\ c\ e)(b\ d\ f)$, σ_1 induces $(a\ d)(b\ e)(c\ f)$, and σ_∞ induces $(a\ b\ c\ d\ e\ f)$. Since the group generated by these three permutations is the abelianization of G, it follows that H is the commutator subgroup of G, a subgroup of index 6, level 6, and genus 1.

4. Remarks and acknowledgements

Several other results similar to the ones discussed here may be found in various places in the literature. For instance, the intersection of two subgroups (smallest common cover of two dessins) may be computed using the *parallel product*; see Wilson [12]. Also, an efficient algorithm for computing the automorphism group of a dessin (centralizer of a transitive subgroup of the symmetric group) was devised by Kuhn [6, Sect. II]), and can be found in GAP [9] as the undocumented function `CentralizerTransSymmCSPG`.

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References

[1] M. Aschbacher, *Finite group theory*, Cambridge Univ. Press, 1986.
[2] B. Birch, *Noncongruence subgroups, covers, and drawings*, In Schneps [8], pp. 25–46.
[3] P. Crawley and R. P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Englewood Cliffs, 1973.
[4] T. Hsu, *Identifying congruence subgroups of the modular group*, Proc. AMS **124** (1996), no. 5, 1351–1359.
[5] G. A. Jones and D. Singerman, *Maps, hypermaps and triangle groups*, In Schneps [8], pp. 115–145.
[6] H. W. Kuhn, *On imprimitive substitution groups*, Amer. J. Math. **26** (1904), 45–102.
[7] L. Schneps, *Dessins d'enfants on the Riemann sphere*, In *The Grothendieck Theory of Dessins d'Enfants* [8], pp. 47–77.
[8] L. Schneps (ed.), *The Grothendieck theory of dessins d'enfants*, Cambridge Univ. Press, 1994.
[9] M. Schönert et al., *GAP: Groups, algorithms and programming*, Lehrstuhl D für Mathematik, RWTH Aachen, April 1992, Version 3.1.
[10] C. C. Sims, *Computation with finitely presented groups*, Cambridge Univ. Press, 1994.
[11] H. Wielandt, *Finite permutation groups*, Academic Press, 1964.
[12] S. E. Wilson, *Parallel products in groups and maps*, J. Alg. **167** (1994), 539–546.
[13] K. Wohlfahrt, *An extension of F. Klein's level concept*, Ill. J. Math. **8** (1964), 529–535.

Department of Mathematics, Univ. of Michigan, Ann Arbor, MI 48109
E-mail address: `timhsu@math.lsa.umich.edu`