Embedding theorems for non-positively curved polygons of finite groups [1]

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The fundamental groups of *complete squared complexes* are a class of groups, some of which are not residually finite. A method is given for embedding the fundamental group of a complete squared complex as a subgroup of a square of finite groups, all of whose (Gersten-Stallings) vertex angles are $\leq \pi/2$. It is also shown that every square of finite groups, all of whose vertex angles are $\leq \pi/2$, can be embedded in a non-positively curved triangle of finite groups. In this way, a non-residually finite, non-positively curved triangle of finite groups is obtained.

1 Overview

It has been known for a long time that the free product of two finite groups amalgamated along a subgroup is residually finite (in fact, virtually free). From a geometrical point of view, an amalgamated free product of two finite groups is just an edge of finite groups in the sense of Bass and Serre. Here, a 'vertex' group is associated to each 0-cell of the edge, and an 'edge' group is associated with the 1-cell of the edge. There is an inclusion of the edge group in each of the vertex groups. The amalgamated free product is then just the free product of the vertex groups, with their edge subgroups identified.

In the 2-dimensional generalization, a group is associated with each cell of a polygon. There is an inclusion of the face group in each edge group and an

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inclusion of each edge group in its associated vertex groups. We assume that at each vertex the intersection of the two edge groups is just the face group. The fundamental group of this polygon of groups is defined to be the free product of the vertex groups amalgamated along the edge groups.

It is natural to ask whether a polygon of finite groups is residually finite. An example where this is not the case is due to K. Brown [3], who showed that Thompson's infinite simple group is actually the fundamental group of a triangle of finite groups.

While in dimension 1 the vertex groups always embed in the amalgamated free product, this does not always hold in dimension 2. A sufficient condition for it to hold was given by Gersten and Stallings [16]. They define an angle between the two edge groups in the vertex group (relative to the face group). In this way an angle is assigned to each corner of the polygon of groups. Their main result is that if the (angled) polygon thus obtained is non-positively curved then the vertex groups embed in the amalgamated free product. So for instance, if the polygon is a triangle then the vertex groups embed provided the sum of the angles is $\leq \pi$. The example of Brown's mentioned above is a triangle of groups with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ and so this triangle of groups is positively curved.

Non-positively curved polygons of finite groups have several very nice properties: They act properly discontinuously and cocompactly on a CAT(0) space with fundamental domain a polygon. Also, Floyd and Parry [5] showed that non-positively curved triangles of groups are automatic. The more stringent class of negatively curved polygons of groups are word-hyperbolic as well [16]. Meier [11,12] has studied the endomorphisms of fundamental groups of polygons of finite groups and has shown that under certain hypotheses their fundamental groups are Hopfian and have outer automorphism groups isomorphic to the groups of (algebraic) symmetries of their underlying polygons of groups.

The main goal of this paper is to show that there exist non-positively curved polygons of finite groups which are not residually finite. To put this question in context, we note that, as mentioned above, negatively curved polygons of groups are word-hyperbolic, and it is still unknown whether all word-hyperbolic groups are residually finite (Gromov [6]). Also, it is interesting to ask whether there is an example of a non-positively curved polygon of finite groups which is not virtually torsion-free. This question can be traced back to H. Neumann [13]. See also Section 8.

An interesting family of four examples of non-positively curved triangles of finite groups was constructed by Tits [17]. (See also Ronan [14], pp. 47–49.) These examples have the same vertex groups and edge groups and only differ in the way in which the edge groups embed as subgroups in the vertex groups.

Two of these are known to be linear, whereas the other two are not known to be residually finite.

Positive results about the residual finiteness of certain non-positively curved squares of (polycyclic by) finite groups have been obtained by G. Kim [8], who showed that if the edge groups are contained in the centers of the associated vertex groups then the group is residually finite. Other positive results concerning the case where the vertex groups are nilpotent and the edge groups are cyclic can be found in Allenby and Tang [1].

Having covered the background of the problem, we now present a brief overview of the results obtained. We begin by discussing the geometric objects known as *complete squared complexes* (Section 2), sometimes abbreviated as *CSCs*. These are 2-complexes whose universal covers are isomorphic to the direct product of two trees. Examples are given in Wise [18] of compact CSCs with non-residually finite fundamental groups.

For every compact CSC X and each prime p, we construct a right angled *non-positively curved square of groups* $R_p(X)$ whose vertex groups are finite p-groups of class 2, and whose edge groups are elementary abelian (Sections 3 and 4).

Our first main result (proved in Section 5) can then be stated as:

Theorem 1.1 (First Main Theorem) *Let X be a compact CSC, and let p be a prime. Then $\pi_1(X)$ can be embedded in $\pi_1(R_p(X))$.*

Unfortunately, the 'smallest' non-residually finite square of finite groups that we obtain in this way has vertex groups of orders 2^{144}, 2^{78}, 2^{62}, 2^{80}, and edge groups of orders 2^6, 2^8, 2^{24}, 2^{24}. Since the examples are a bit unwieldy, we have chosen not to write out presentations for them, but the interested reader could apply our construction to a CSC from [18].

In order to provide examples of non-positively curved triangles of finite groups which are not residually finite, we also prove another embedding theorem of independent interest (Sections 6 and 7).

Theorem 1.2 (Second Main Theorem) *Let R be a square of finite groups, all of whose vertex angles are $\leq \pi/2$. Then $\pi_1(R)$ can be embedded in the fundamental group of a non-positively curved triangle of finite groups.*

Our result about residual finiteness (Corollary 1.4) is then obtained from the following.

Theorem 1.3 *There exists a compact CSC whose fundamental group is not residually finite.*

PROOF. See [18].

Using the Main Theorems, we can embed the group of Theorem 1.3 into either a non-positively curved square or a non-positively curved triangle of finite groups. Since subgroups of residually finite groups are themselves residually finite, the chosen square or triangle of finite groups has non-residually finite fundamental group. In other words:

Corollary 1.4 *There exists a non-positively curved triangle (resp. square) of finite groups with non-residually finite fundamental group.*

Incidentally, since it is not difficult to construct a square of finite groups whose fundamental group contains $F_2 \times F_2$ as a subgroup and whose vertex angles are all $\pi/2$, the Second Main Theorem also implies that there exists a non-positively curved triangle of finite groups with incoherent fundamental group.

2 Complete squared complexes

In this section, we define CSCs and summarize some of their relevant properties, as proved in [18].

Recall that a *complete bipartite graph* is a graph Γ whose vertices partition into two sets V_1 and V_2 such that Γ has exactly one edge between every vertex in V_1 and every vertex in V_2, and no others. In particular, a complete bipartite graph has no self-loops and no multiple edges.

Definition 2.1 Let X be a combinatorial 2-complex whose (closed) 2-cells are "squares". We say that X is a *complete squared complex* (or *CSC*) provided that the edges of $X^{(1)}$ are partitioned into 2 classes in such a way that for each $v \in X^{(0)}$, the corresponding partition of the 0-cells of the graph $\mathrm{Link}(v)$ induces a complete bipartite structure on the graph $\mathrm{Link}(v)$.

The above definition is almost the same as the simpler requirement that the link of each vertex of X be a complete bipartite graph. The difference is that the above definition imposes a vertical-horizontal structure on X. A complex which satisfies only the complete bipartite graph condition has a double cover which is a CSC as defined above. See [18].

For the rest of this section, let X be a CSC, and let F be the set of 2-cells of X. Our next goal is to label the 1-cells of X appropriately. Let BX denote the *barycentric subdivision* of X. The following is then easy to see.

Theorem 2.2 *If X is a CSC, then BX is a CSC homeomorphic to X.*

Let I^2 be the closed unit square in $\mathbf{R}^2$.

Theorem 2.3 *There exists a continuous combinatorial map from $BX \mapsto I^2$.*

PROOF. There are obvious combinatorial maps from the subdivision of each square of X to I^2. It is easy to see that one can choose one such map for each subdivided square so that the maps agree on $BX^{(1)}$. Note that Theorem 2.3 does not really need X to be a CSC; in fact, it is sufficient that X be a VH-complex, that is, that the edges of X are partitioned into two classes in a manner respected by the attaching maps of squares. (See [18].)

Definition 2.4 Let X be a CSC equipped with a map like the one in Theorem 2.3. If we mark the edges of I^2 as being north, south, east and west in the obvious way, the edges of X divide into classes N, S, E and W, corresponding to the preimages of the appropriate edges of I^2. The N and S edges make up what we call the *horizontal* 1-skeleton of X, and the E and W edges make up the *vertical* 1-skeleton of X. Furthermore, suppose the edges of I^2 are oriented clockwise; that is, the north edge is oriented to the right, the east edge down, the south edge to the left, and the west edge up. Pulling back this orientation, we get an orientation of the edges of X.

We call the above arrangement (labels, horizontal/vertical 1-skeletons, and orientation) an *NESW labelling* for X. Since barycentric subdivision does not alter the topology of a CSC (Theorem 2.2), without losing any generality for the proof of the First Main Theorem (Theorem 1.1), we may freely assume that any CSC is NESW-labelled.

Remark 2.5 Let X be an NESW-labelled CSC. The link condition in Definition 2.1 implies that any two edges from distinct NESW classes intersecting at a vertex of X determine a unique face of X. For instance, a face f can be specified by naming its north and west edges. We will therefore sometimes refer to a face of X by listing its NESW-labelled edges, e.g., $f = (n, e, s, w)$.

If X is a combinatorial complex, we denote the fundamental groupoid of X on the basepoints $X^{(0)}$ by $\pi(X)$. If X is an NESW-labelled CSC, repeated application of the van Kampen theorem for the fundamental groupoid (R. Brown [4, 8.4]) gives us the following description of $\pi(X)$.

Theorem 2.6 *$\pi(X)$ has a (groupoid) presentation with generators $n \in N$, $s \in S$, $e \in E$, and $w \in W$, corresponding to the edges of X, and relations*

$$nesw = 1 \tag{2.1}$$

for all $(n, e, s, w) = f$ such that $n \in N$, $e \in E$, $s \in S$, $w \in W$, and $f \in F$.

Besides the NESW labelling, the property of CSCs which is most important to us is the following.

Theorem 2.7 *If X is a CSC then $\tilde{X}$, the universal cover of X, is isomorphic (as a complex) to the product of a vertical tree and a horizontal tree.*

PROOF. See [18].

Corollary 2.8 *Let X be a CSC, let x_0 be a point in $X^{(0)}$, and let σ be a path representing an element of $\pi_1(X, x_0)$. Then σ is homotopic to the composition $\sigma_v \circ \sigma_h$, where σ_v (resp. σ_h) is a combinatorial path travelling in the vertical (resp. horizontal) 1-skeleton of X.*

PROOF. Consider the lift of σ to $\tilde{X}$ at some basepoint for $\tilde{X}$. By Theorem 2.7, there exists a path in $\tilde{X}$ connecting the endpoints of σ which travels first horizontally and then vertically.

3 Construction of the vertex groups

In this section, we fix a prime p, finite indexing sets I and J, and a subset F of $I \times J$. We also define the commutator $[a, b]$ to be $aba^{-1}b^{-1}$.

Our goal in this section is to construct a finite p-group $Q_{IJF}(p)$ with the properties listed in Theorem 3.2. We define $Q_{IJF}(p)$ to be the group with generators x_i (for all $i \in I$), ξ_j (for all $j \in J$), y_f (for all $f \in F$), and ψ_f (for all $f \in F$); and defining relations

$$1 = x_i^p = \xi_j^p = y_f^p = \psi_f^p \tag{3.1}$$
$$1 = [x_i, x_k] = [\xi_j, \xi_l] = [y_f, y_g] = [\psi_f, \psi_g] \tag{3.2}$$
$$1 = [x_i, y_f] = [x_i, \psi_f] = [\xi_j, y_f] = [\xi_j, \psi_f] = [y_f, \psi_g] \tag{3.3}$$
$$[x_i, \xi_j] = \begin{cases} y_f\psi_f & (i, j) = f \in F \\ 1 & (i, j) \notin F \end{cases} \tag{3.4}$$

for all $i, k \in I$, $j, l \in J$, $f, g \in F$. (In other words, all the generators have order p, and all the generators commute, except for "incident" x_i and ξ_j.) We also define the subgroups $X_{IJF}(p) = \langle x_i \rangle$, $\Xi_{IJF}(p) = \langle \xi_j \rangle$, $Y_{IJF}(p) = \langle y_f \rangle$, and $\Psi_{IJF}(p) = \langle \psi_f \rangle$ of $Q_{IJF}(p)$.

Theorem 3.1 *$X_{IJF}(p)$, $\Xi_{IJF}(p)$, $Y_{IJF}(p)$, and $\Psi_{IJF}(p)$ are elementary abelian p-groups of rank $|I|$, $|J|$, $|F|$, and $|F|$, respectively; and any element*

$q \in Q_{IJF}(p)$ has a unique normal form

$$q = x\xi y\psi \tag{3.5}$$

for some $x \in X_{IJF}(p)$, $\xi \in \Xi_{IJF}(p)$, $y \in Y_{IJF}(p)$, and $\psi \in \Psi_{IJF}(p)$. In particular, $Q_{IJF}(p)$ is a finite p-group.

PROOF. This follows from the normal form theorem for polycyclic presentations. See Sims [15, 9.4].

For all $(i,j) = f \in F$, we define $x_f = x_i y_f$, $\xi_f = \xi_j \psi_f$, $T_{IJF}(p) = \langle x_f \rangle$, and $\Theta_{IJF}(p) = \langle \xi_f \rangle$.

Theorem 3.2 *$Q_{IJF}(p)$ and its subgroups have the following properties:*

(i) For all $(i,j) = f \in F$,

$$x_i \xi_j x_f^{-1} \xi_f^{-1} = 1. \tag{3.6}$$

(ii) We have

$$(X_{IJF}(p) Y_{IJF}(p)) \cap (\Xi_{IJF}(p) \Psi_{IJF}(p)) = 1. \tag{3.7}$$

In particular, $T_{IJF}(p) \cap \Theta_{IJF}(p) = 1$.

(iii) $T_{IJF}(p)$ and $\Theta_{IJF}(p)$ are elementary abelian of rank $|F|$.

PROOF. First, property (i) results from rewriting (3.4):

$$\begin{aligned} x_i \xi_j x_i^{-1} \xi_j^{-1} &= y_f \psi_f \\ x_i \xi_j y_f^{-1} x_i^{-1} \psi_f^{-1} \xi_j^{-1} &= 1 \\ x_i \xi_j (x_i y_f)^{-1} (\xi_j \psi_f)^{-1} &= 1 \\ x_i \xi_j x_f^{-1} \xi_f^{-1} &= 1. \end{aligned} \tag{3.8}$$

Next, property (ii) follows from the normal form (3.5). Finally, property (iii) is obtained by noting that the elementary abelian subgroups $X_{IJF}(p) Y_{IJF}(p)$ and $T_{IJF}(p) X_{IJF}(p)$ (resp. $\Xi_{IJF}(p) \Psi_{IJF}(p)$ and $\Theta_{IJF}(p) \Xi_{IJF}(p)$) are equal, and comparing ranks.

Remark 3.3 We note that p being prime is basically irrelevant to the above construction; it simply makes the result easier to describe. In fact, the initial version of the construction was done only for $p = 2$. We have included a more general version only to assure the reader that no special properties of 2-groups are being used.

Remark 3.4 As $T_{IJF}(p)$ and $\Theta_{IJF}(p)$ will be used as edge groups in the next section, it is worth remarking that neither group is central in $Q_{IJF}(p)$, unless F is the empty set. Compare Kim [8].

4 Construction of $R_p(X)$

In this section, we consider squares of groups with trivial 2-cell groups, or in other words, colimits of diagrams of inclusions like the following:

$$\begin{array}{ccccc} A & \leftarrow & E_{AB} & \rightarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ E_{DA} & \leftarrow & 1 & \rightarrow & E_{BC} \\ \downarrow & & \downarrow & & \downarrow \\ D & \leftarrow & E_{CD} & \rightarrow & C \end{array} \qquad (4.1)$$

We set some terminology. In the square (4.1), the groups A, B, C, and D are called the *vertex groups*; the groups E_{AB}, E_{BC}, E_{CD}, and E_{DA} are called the *edge groups*; and the amalgams $A *_{E_{AB}} B$, $B *_{E_{BC}} C$, $C *_{E_{CD}} D$, and $D *_{E_{DA}} A$ are called the *edge amalgams*.

Definition 4.1 Let X be a compact NESW-labelled CSC, and let p be a prime. We define the square of groups $R_p(X)$ by defining the vertex and edge groups in (4.1).

First, we note that since any two differently labelled intersecting edges of X determine a unique face of X (Remark 2.5), the map sending $(n, e, s, w) = f \in F$ to (w, n) (resp. (n, e), (e, s), (s, w)) is a bijective correspondence between F and a subset of $W \times N$ (resp. $N \times E$, $E \times S$, $S \times W$). Therefore, the group $Q_{WNF}(p)$ (resp. $Q_{NEF}(p)$, $Q_{ESF}(p)$, $Q_{SWF}(p)$) is well-defined. We set:

(i) $A \cong Q_{WNF}(p)$, with x and ξ replaced by a and α. That is, A is generated by a_w, α_n, a_f, and α_f (for all $w \in W$, $n \in N$, $f \in F$); and (3.6) becomes

$$a_w \alpha_n a_f^{-1} \alpha_f^{-1} = 1. \qquad (4.2)$$

(ii) $B \cong Q_{NEF}(p)$, with x and ξ replaced by b and β. For $n \in N$ and $e \in E$, (3.6) becomes

$$b_n \beta_e b_f^{-1} \beta_f^{-1} = 1. \qquad (4.3)$$

(iii) $C \cong Q_{ESF}(p)$, with x and ξ replaced by c and γ. (3.6) becomes

$$c_e \gamma_s c_f^{-1} \gamma_f^{-1} = 1. \qquad (4.4)$$

(iv) $D \cong Q_{SWF}(p)$, with x and ξ replaced by d and δ. (3.6) becomes

$$d_s \delta_w d_f^{-1} \delta_f^{-1} = 1. \tag{4.5}$$

The edge groups are all elementary abelian p-groups of rank $|F|$. Instead of naming the elements of each edge group and specifying embeddings into the vertex groups, we will just indicate an isomorphism between subgroups of the respective vertex groups by forming a one-to-one correspondence between generators of an elementary abelian p-group on each side.

$$E_{AB} : \left\{ a_f \leftrightarrow \beta_f^{-1} \quad \text{for all } f \in F \right\} \tag{4.6}$$
$$E_{BC} : \left\{ b_f \leftrightarrow \gamma_f^{-1} \quad \text{for all } f \in F \right\} \tag{4.7}$$
$$E_{CD} : \left\{ c_f \leftrightarrow \delta_f^{-1} \quad \text{for all } f \in F \right\} \tag{4.8}$$
$$E_{DA} : \left\{ d_f \leftrightarrow \alpha_f^{-1} \quad \text{for all } f \in F \right\} \tag{4.9}$$

We note that if the edge group identifications are considered as relations in $\pi_1(R_p(X))$, then we have:

$$a_f \beta_f = 1 \tag{4.10}$$
$$b_f \gamma_f = 1 \tag{4.11}$$
$$c_f \delta_f = 1 \tag{4.12}$$
$$d_f \alpha_f = 1 \tag{4.13}$$

We also note that the edge groups contained in a vertex group embed in exactly the same way that $T_{IJF}(p)$ and $\Theta_{IJF}(p)$ embed in $Q_{IJF}(p)$ (Section 3).

Some key properties of $R_p(X)$ are collected together in the following theorem.

Theorem 4.2 *$R_p(X)$ has the following properties:*

(i) All of the vertex angles of $R_p(X)$ are $\leq \pi/2$; in particular, $R_p(X)$ is non-positively curved.
(ii) The edge amalgams of $R_p(X)$ inject into $\pi_1(R_p(X))$.
(iii) The intersection in $\pi_1(R_p(X))$ of the edge amalgams of two intersecting edges of $R_p(X)$ is precisely the vertex group of their intersection. For instance,

$$(A *_{E_{AB}} B) \cap (B *_{E_{BC}} C) = B \tag{4.14}$$

in $\pi_1(R_p(X))$.
(iv) Every non-empty reduced word in the generators $\omega_n = \alpha_n b_n$ (resp. $\omega_e = \beta_e c_e$) is equal to a reduced word of length at least 2 in $A *_{E_{AB}} B$ (resp. $B *_{E_{BC}} C$).

PROOF. Property (i) follows from property (ii) of Theorem 3.2.

Properties (ii) and (iii) are easy to see geometrically. Consider, without loss of generality, the north and east edges of the square. Property (ii) follows because the trees corresponding to the north and east edge amalgams embed equivariantly (as convex sub-orbihedra) in the universal covering orbihedron of $R_p(X)$. It follows that the intersection of the two subgroups is precisely the subgroup stabilizing *both* of the trees, and therefore, their intersection. However, the two trees intersect in the vertex stabilized by B, and so the intersection of the two subgroups is B, which gives us (iii).

Finally, as for property (iv), it is enough to consider the $A *_{E_{AB}} B$ case. By the normal form theorem for free products, it suffices to show that the subgroup $\langle \alpha_n, b_n \rangle \leq A *_{E_{AB}} B$ is the free product of $\langle \alpha_n \rangle \leq A$ and $\langle b_n \rangle \leq B$. However, by the normal form theorem for free products with amalgamation (Lyndon and Schupp [9]), it is enough to show that $\langle \alpha_n \rangle \cap \langle a_f \rangle = \langle b_n \rangle \cap \langle \beta_f \rangle = 1$, and this follows from property (ii) of Theorem 3.2.

Remark 4.3 Alternately, properties (ii) and (iii) of Theorem 4.2 follow easily from the amalgam decompositions

$$\pi_1(R_p(X)) = (A *_{E_{AB}} B) *_{(E_{DA} * E_{BC})} (D *_{E_{DC}} C) \tag{4.15}$$

and

$$\pi_1(R_p(X)) = (A *_{E_{DA}} D) *_{(E_{AB} * E_{CD})} (B *_{E_{BC}} C), \tag{4.16}$$

together with the normal form theorem for free products with amalgamation.

5 Proof of the First Main Theorem

Let X be a compact NESW-labelled CSC, and let p be a prime. We wish to define a homomorphism of groupoids φ from the fundamental groupoid of X to $\pi_1(R_p(X))$ by the following rules.

$$\begin{aligned} \varphi(n) &= \alpha_n b_n && \text{for all } n \in N \\ \varphi(e) &= \beta_e c_e && \text{for all } e \in E \\ \varphi(s) &= \gamma_s d_s && \text{for all } s \in S \\ \varphi(w) &= \delta_w a_w && \text{for all } w \in W \end{aligned} \tag{5.1}$$

The First Main Theorem is then a consequence of the following theorem.

Theorem 5.1 *We have:*

(i) The rules in (5.1) *determine a homomorphism of groupoids φ from the fundamental groupoid of X to $\pi_1(R_p(X))$.*
(ii) The restriction of φ to $\pi_1(X, x_0)$ is injective.

PROOF. From Theorem 2.6, for assertion (i) to hold, it suffices to show that for all $(n, e, s, w) = f \in F$,

$$\varphi(nesw) = \alpha_n b_n \beta_e c_e \gamma_s d_s \delta_w a_w = 1 \tag{5.2}$$

in $\pi_1(R_p(X))$. However, (5.2) follows from the Van Kampen diagram in Figure 1. Vertices in Figure 1 are represented by solid black squares, and each relation polygon is marked with the number of the equation which implies it.

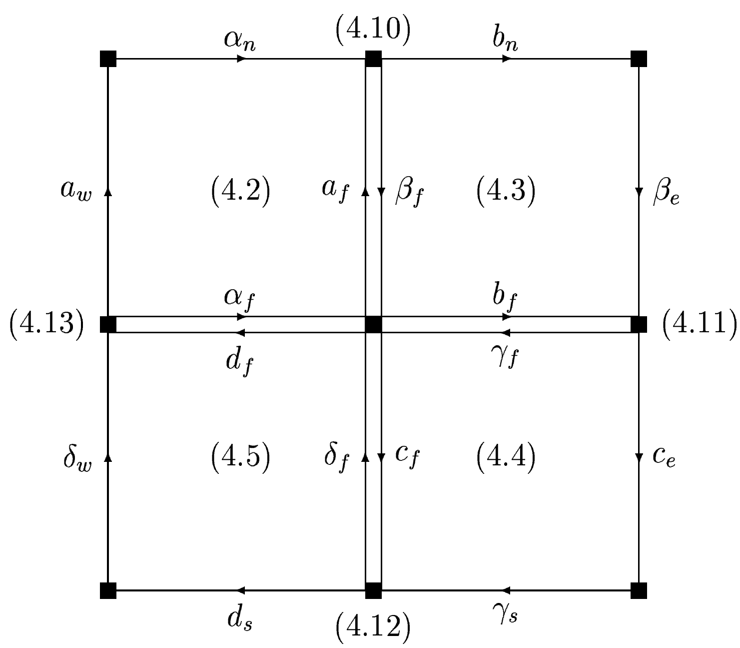

Fig. 1. Van Kampen diagram for proof of First Main Theorem

It remains to prove assertion (ii). Choose $x_0 \in X^{(0)}$ to be a "northeast" vertex in terms of the NESW labelling. Let V be the connected component of the vertical 1-skeleton of X containing x_0, and let H be the connected component of the horizontal 1-skeleton of X containing x_0. Note that V consists of only east edges, and H consists of only north edges.

Now, the composition of φ with the natural homomorphism induced by the inclusion of V (resp. H) in X defines a homomorphism from $\pi(V)$ (resp. $\pi(H)$) to $\pi_1(R_p(X))$. Both of these homomorphisms are given by the rules (5.1), so by abuse of notation, we will call them both φ as well. Further inspection of (5.1) shows that φ maps $\pi(V)$ (resp. $\pi(H)$) into the edge amalgam $B *_{E_{BC}} C$ (resp. $A *_{E_{AB}} B$), considered as a subgroup of $\pi_1(R_p(X))$ (Theorem 4.2, property ii). In fact, φ sends non-trivial reduced paths in V (resp. H) to non-empty reduced words in the generators $\omega_e = \beta_e c_e$ (resp. $\omega_n = \alpha_n b_n$), which, by Theorem 4.2, property iv, are equal to words of reduced length at least 2 in $B *_{E_{BC}} C$ (resp. $A *_{E_{AB}} B$).

So now suppose that $\sigma \in \pi_1(X, x_0)$ is in the kernel of φ. To prove assertion (ii), we need to show that $\sigma = 1$. From Corollary 2.8, σ is homotopic to the composition $\sigma_v \circ \sigma_h$, where σ_v (resp. σ_h) is the image under inclusion of a path $v \in \pi(V)$ (resp. $h \in \pi(H)$). Therefore, since

$$1 = \varphi(\sigma) = \varphi(v)\varphi(h), \tag{5.3}$$

the element $\varphi(v) \in (B *_{E_{BC}} C)$ must be equal to $\varphi(h)^{-1} \in (A *_{E_{AB}} B)$ in $\pi_1(R_p(X))$. Therefore, by property (iii) of Theorem 4.2,

$$\varphi(v) = \varphi(h)^{-1} \in (A *_{E_{AB}} B) \cap (B *_{E_{BC}} C) = B. \tag{5.4}$$

However, since φ sends non-trivial reduced paths in V (resp. H) to reduced words in the east (resp. north) edge amalgam of length at least 2, v and h must be trivial. Therefore, σ_v and σ_h are both trivial, and $\sigma = 1$.

6 Unfolding dihedral triangles

The idea behind the Second Main Theorem comes from looking at Coxeter groups. Let $\pi_1(P)$ be the Coxeter group generated by the reflections in the walls of the square tiling of the plane, and let $\pi_1(T)$ be the $(2,4,4)$ triangle group corresponding to the subdivision of the square tiling into $(\pi/2, \pi/4, \pi/4)$ triangles, as shown in Figure 2. (The reason for our notation will become clear later.) In fact, Figure 2 shows that $\pi_1(T)$ is the semidirect product of $\pi_1(P)$ and the dihedral group of order 8 generated by the reflections s and t.

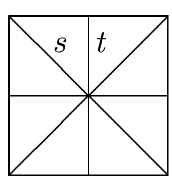

Fig. 2. A square group inside a triangle group

More generally, suppose $\pi_1(P)$ is the discrete group generated by reflections in the tiling of the spherical/Euclidean/hyperbolic plane by regular n-gons with vertex angle θ. Then the n-gons can be subdivided into $(\pi/2, \theta/2, \pi/n)$

triangles, analogously to Figure 2, and the corresponding triangle group $\pi_1(T)$ is the semidirect product of $\pi_1(P)$ and the dihedral group of order $2n$. Our goal (Theorems 6.3 and 7.2) is to show that we can do something similar for *any* n-gon of groups, and not just an n-gon of Coxeter groups.

Of course, the above situation has several special features as an n-gon of groups, namely:

(i) All of the vertex groups (resp. edge groups) are isomorphic.
(ii) In the notation of Figure 2, conjugation by s (resp. t) is an automorphism of order 2 of a vertex (resp. edge) group.

We axiomatize these properties in the following definition.

Definition 6.1 A *dihedral embedding* is an 8-tuple $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$, where n is an integer ≥ 4; $\mathbf{F}$, $\mathbf{E}$, and $\mathbf{V}$ are groups; $\varphi : \mathbf{F} \rightarrowtail \mathbf{E}$ and $\psi : \mathbf{E} \rightarrowtail \mathbf{V}$ are inclusions; and σ (resp. τ) is an automorphism of $\mathbf{V}$ (resp. $\mathbf{E}$) such that $\sigma^2 = 1$ (resp. $\tau^2 = 1$), $\sigma(\mathbf{F}) = \mathbf{F}$ (resp. $\tau(\mathbf{F}) = \mathbf{F}$), and $(\tau\sigma)^n = 1$. (Note that as an automorphism, $(\tau\sigma)^n$ is only defined on $\mathbf{F}$.) We will often omit φ and ψ and simply consider $\mathbf{F}$ to be a subgroup of $\mathbf{E}$ and $\mathbf{E}$ to be a subgroup of $\mathbf{V}$.

Our basic strategy now is to build the triangle that should result from the process described by Figure 2, and then find an n-gon of groups inside it (Theorem 6.3). Let D_{2n} be the dihedral group $\langle s, t | 1 = s^2 = t^2 = (ts)^n \rangle$ of order $2n$. Given a dihedral embedding, we can define the semidirect products $\mathbf{V} \rtimes \langle s \rangle$, $\mathbf{E} \rtimes \langle t \rangle$, $\mathbf{F} \rtimes \langle s \rangle$, $\mathbf{F} \rtimes \langle t \rangle$, and $\mathbf{F} \rtimes D_{2n}$ by declaring that $svs^{-1} = \sigma(v)$ for $v \in \mathbf{V}$, and $tet^{-1} = \tau(e)$ for $e \in \mathbf{E}$.

Definition 6.2 The *triangle of a dihedral embedding* $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ is the triangle of groups given by the diagram of inclusions in Figure 3.

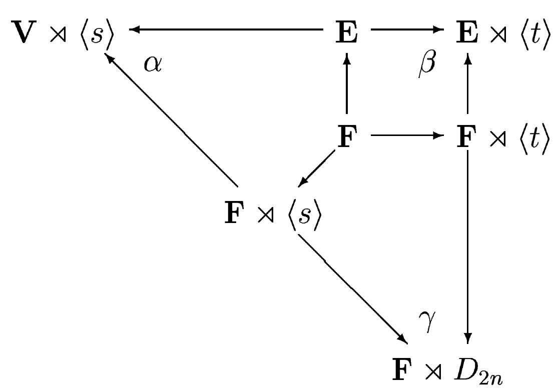

Fig. 3. Triangle of a dihedral embedding

The *n-gon of a dihedral embedding* $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ is the n-gon of groups described as follows:

(i) The vertex groups are all isomorphic to $\mathbf{V}$, the edge groups are all isomorphic to $\mathbf{E}$, and the face group is isomorphic to $\mathbf{F}$;

(ii) The counterclockwise edge inclusion maps (from $\mathbf{E}$ into $\mathbf{V}$) are all $\psi\tau$, and the clockwise edge inclusion maps are all $\sigma\psi$; and

(iii) The face inclusion maps (from $\mathbf{F}$ into $\mathbf{E}$) are, in clockwise order, the maps $\varphi(\tau\sigma)^k$, where k runs over all numbers mod n.

For example, when $n = 4$, the n-gon of a dihedral embedding is the square of groups shown in Figure 4.

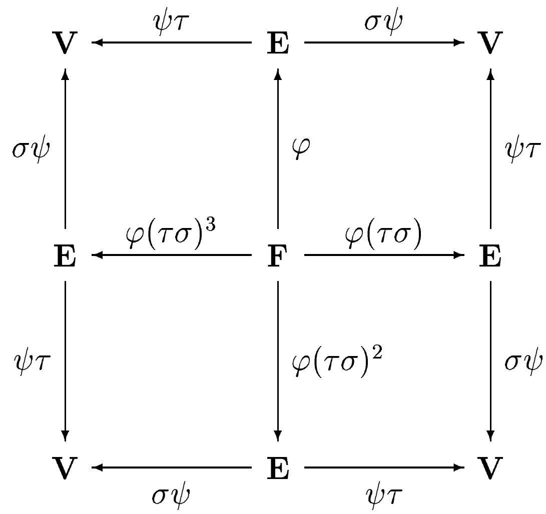

Fig. 4. n-gon of a dihedral embedding, $n = 4$

Note that, by construction, all of the vertex angles of an n-gon of a dihedral embedding are equal to the angle arising from the situation shown in Figure 5. By abuse of notation, we use $\sigma(\mathbf{E}) *_{\mathbf{F}} \mathbf{E}$ to denote the amalgamated free product arising from the lower left portion of Figure 5.

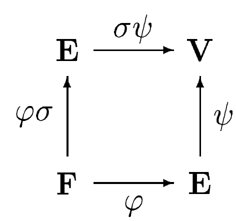

Fig. 5. Typical situation at a vertex angle

Theorem 6.3 *Let $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ be a dihedral embedding, let T be its triangle, and let P be its n-gon. Let α, β, and γ be the vertex angles indicated by Figure 3, and let θ be the vertex angle of the vertices of P. Let $\Phi : \pi_1(T) \mapsto D_{2n}$ be the surjection obtained by killing $\mathbf{V}$, $\mathbf{E}$, and $\mathbf{F}$. Then:*

(i) $\alpha = \theta/2$, $\beta = \pi/2$, *and* $\gamma = \pi/n$.

(ii) $\pi_1(P) \cong \ker \Phi$.

PROOF. The fact that $\beta = \pi/2$ follows because $\mathbf{E} \cap \mathbf{F} \rtimes \langle t \rangle = \mathbf{F}$ inside

$\mathbf{E} \rtimes \langle t \rangle$. Furthermore, since the γ vertex of T is modelled on D_{2n}, it follows easily that $\gamma = \pi/n$. To prove (i), then, it remains to show that $\alpha = \theta/2$.

So let $\theta = \pi/m$. Suppose that

$$w = fse_1se_2 \dots se_l, \tag{6.1}$$

where the e_i are elements of $\mathbf{E}$ not contained in $\mathbf{F}$, is a non-trivial element of length $2l$ of the kernel of the natural map from $(\mathbf{F} \rtimes \langle s \rangle) *_{\mathbf{F}} \mathbf{E}$ to $\mathbf{V} \rtimes \langle s \rangle$. First of all, if l is odd, $\Phi(w) = s$. So l is even, and we have

$$\begin{aligned} w &= fse_1se_2 \dots se_l \\ &= f(se_1s^{-1})e_2(se_3s^{-1})e_4 \dots (se_{l-1}s^{-1})e_l \\ &= f\sigma(e_1)e_2\sigma(e_3)e_4 \dots \sigma(e_{l-1})e_l. \end{aligned} \tag{6.2}$$

Therefore, w also determines a non-trivial element of length l of the kernel of the natural map from $\sigma(\mathbf{E}) *_{\mathbf{F}} \mathbf{E}$ to $\mathbf{V}$, and so, by the definition of vertex angle, we must have $l \geq 2m$. Since this is true for any such w, $\alpha \leq \theta/2$. Finally, running this argument backwards, letting w be a non-trivial element of length $2m$ of the kernel of the natural map from $\sigma(\mathbf{E}) *_{\mathbf{F}} \mathbf{E}$ to $\mathbf{V}$, implies that $\alpha = \theta/2$.

As for (ii), the most natural proof would use covering spaces of orbihedra (see Haefliger [7]). However, to avoid the ensuing technical discussion, we will use the more elementary, if perhaps less appealing, approach of appying the Reidemeister-Schreier method. First, by writing down the colimit of T in terms of generators and relations, we see that $\pi_1(T)$ is given by the following presentation.

$$\begin{aligned} \Big\langle s,\ t,\ v \in \mathbf{V},\ e \in \mathbf{E},\ f \in \mathbf{F}\ \Big|& \\ & 1 = s^2 = t^2 = (ts)^n, && (6.3)\\ & v_1v_2 = v_3,\ e_1e_2 = e_3,\ f_1f_2 = f_3, && (6.4)\\ & tet^{-1} = \tau(e) \qquad \text{for all } e \in \mathbf{E}, && (6.5)\\ & svs^{-1} = \sigma(v) \qquad \text{for all } v \in \mathbf{V}, && (6.6)\\ & f = \varphi(f) \qquad \text{for all } f \in \mathbf{F}, && (6.7)\\ & e = \psi(e) \qquad \text{for all } e \in \mathbf{E} \Big\rangle && (6.8) \end{aligned}$$

In other words, the generators are s, t, and letters corresponding to the elements of $\mathbf{V}$, $\mathbf{E}$, and $\mathbf{F}$, with defining relations (6.3)–(6.8). The relations (6.4) are meant to be the "multiplication table" relations in $\mathbf{V}$, $\mathbf{E}$, and $\mathbf{F}$; that is, the v relation is taken over all v_1, v_2, v_3 such that $v_1v_2 = v_3$, and the e and f relations work similarly.

Let $K = \ker \Phi$. We apply the Reidemeister-Schreier method (Magnus, Karrass, and Solitar [10, 2.3]), enumerating the right cosets of K, and taking 1, t, ts, tst, $\ldots$ $(ts)^{n-1}t$ as our Schreier system. The result is the following presentation for K.

Generators. Reidemeister-Schreier gives us two classes of generators for K.

(i) The elements $v_d = dvd^{-1}$, $e_d = ded^{-1}$, and $f_d = dfd^{-1}$, where v, e, f, and d run over all elements of $\mathbf{V}$, $\mathbf{E}$, $\mathbf{F}$, and the Schreier system, respectively. We define $\mathbf{V}_d = \langle v_d \rangle$, $\mathbf{E}_d = \langle e_d \rangle$, and $\mathbf{F}_d = \langle f_d \rangle$.
(ii) Words w_i in s and t which define trivial elements of D_{2n}.

Relations. A set of defining relations for K is obtained by rewriting the relations $\{dr_i d^{-1}\}$ in terms of the above generators, where d runs over all elements of the Schreier system, and r_i is a relation in (6.3)–(6.8). We have:

(6.3) These relations kill the w_i's, and allow us to consider the d's in v_d, $\mathbf{V}_d$, etc., to be elements of D_{2n}, and not just elements of the Schreier system.
(6.4) These relations say precisely that $\mathbf{V}_d$, $\mathbf{E}_d$, and $\mathbf{F}_d$ are copies of the groups $\mathbf{V}$, $\mathbf{E}$, and $\mathbf{F}$, respectively. Note that this means that by ignoring subscripts, we can define automorphisms $\sigma : \mathbf{V}_d \mapsto \mathbf{V}_d$, $\tau : \mathbf{E}_d \mapsto \mathbf{E}_d$, etc., and isomorphisms $\sigma : \mathbf{V}_c \mapsto \mathbf{V}_d$, $\tau : \mathbf{E}_c \mapsto \mathbf{E}_d$, etc., for all $c, d \in D_{2n}$. Similarly, we can define embeddings $\varphi : \mathbf{F}_d \mapsto \mathbf{E}_d$ and $\psi : \mathbf{E}_d \mapsto \mathbf{V}_d$.
(6.5) For all $d \in D_{2n}$ and $e \in \mathbf{E}$, the relation

$$(dt)e(t^{-1}d^{-1}) = d\tau(e)d^{-1} \tag{6.9}$$

can be rewritten as

$$e_{dt} = \tau(e_d). \tag{6.10}$$

In other words, we identify $\mathbf{E}_d$ with $\mathbf{E}_{dt}$ by the isomorphism τ. (Note that the t accumulates on the right, even though τ is a left automorphism.)
(6.6) Analogously to (6.5), we rewrite these relations by saying that for all $d \in D_{2n}$, we identify $\mathbf{V}_d$ with $\mathbf{V}_{ds}$ by the isomorphism σ.
(6.7) Again analogously to (6.5), we rewrite these relations by saying that for all $d \in D_{2n}$, we identify $\mathbf{F}_d$ with a subgroup of $\mathbf{E}_d$ by the embedding φ.
(6.8) We rewrite these relations by saying that for all $d \in D_{2n}$, we identify $\mathbf{E}_d$ with a subgroup of $\mathbf{V}_d$ by the embedding ψ.

For notational and typographical expediency, in the remainder of the proof, we assume $n = 4$, as the details are essentially the same for $n > 4$. For $n = 4$,

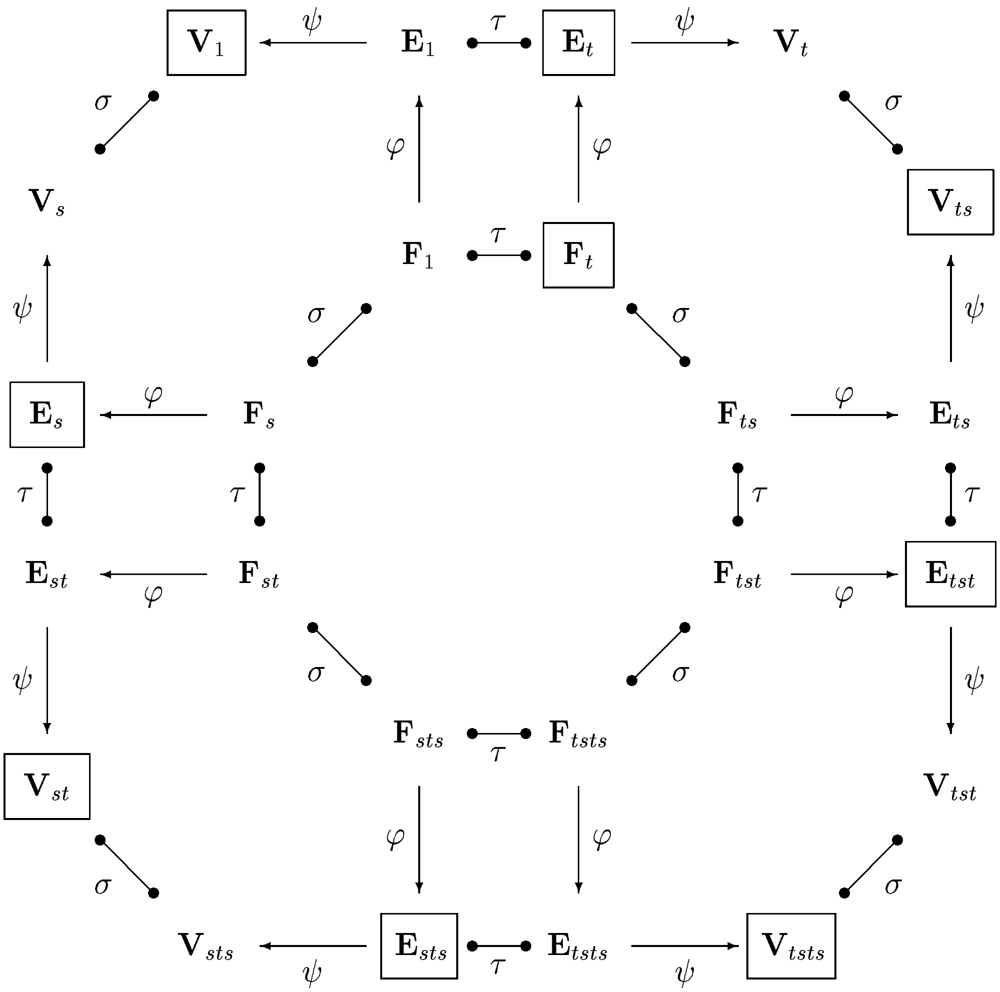

Fig. 6. Amalgam used in proof of Theorem 6.3

inspection of **(6.5)**–**(6.8)** shows that the above presentation for K is precisely the amalgam pictured in Figure 6. (Note that we use $tsts = stst$ to close up the bottom of the diagram.) We may simplify this amalgam by choosing a single "representative" in each "component" of isomorphisms of type σ or τ. (These "components" are indicated in Figure 6 by lines with dots on either end.) In particular, if we choose the "representatives" boxed in Figure 6, and we discard the subscripts, a diagram chase shows that Figure 6 simplifies to Figure 4, which means that $K \cong \pi_1(P)$.

7 Proof of the Second Main Theorem

Having shown that a "regular" polygon of groups can be embedded in an appropriate triangle of groups (Theorem 6.3), we now must show than an arbitrary polygon of groups can be embedded in an appropriate regular polygon of groups. There are many ways of doing this, but the following method (resulting in Theorem 7.2) is one which suits our purposes here. (See Remark 7.7, though.)

Hereafter, by convention, we write the direct product of n groups $G_1, \ldots, G_n$ as $(G_1, \ldots, G_n)$, thinking of the n-tuple as coordinates. We follow the same convention for elements of a direct product; for instance, we write the identity in the above group as $(1, \ldots, 1)$.

Throughout this section, let P be an n-gon of groups with face group F, edge groups $E_1, \ldots, E_n$, and vertex groups $V_1, \ldots, V_n$ such that the edge group E_i is included in the vertex groups V_i and V_{i+1}, where $i+1$ is considered (mod n). We will also assume throughout that at least one of $E_1, \ldots, E_n$ contains F as a proper subgroup. (If this is not true, then $\pi_1(P)$ is an amalgamated free product of $V_1, \ldots, V_n$ over F, so we lose little with this assumption.) For example, when $n = 4$, we have:

$$\begin{array}{ccccc} V_1 & \leftarrow & E_1 & \rightarrow & V_2 \\ \uparrow & & \uparrow & & \uparrow \\ E_4 & \leftarrow & F & \rightarrow & E_2 \\ \downarrow & & \downarrow & & \downarrow \\ V_4 & \leftarrow & E_3 & \rightarrow & V_3 \end{array} \tag{7.1}$$

Let $\mathbf{V} = (V_1, V_1, V_2, V_2, \ldots, V_n, V_n)$, $\mathbf{E} = (E_n, E_1, E_1, E_2, \ldots, E_{n-1}, E_n)$, and $\mathbf{F} = (F, F, F, F, \ldots, F, F)$ (i.e., let $\mathbf{F} = F^{2n}$). Let $\varphi : \mathbf{F} \mapsto \mathbf{E}$ and $\psi : \mathbf{E} \mapsto \mathbf{V}$ be injections defined by applying the appropriate injections of P coordinatewise. Finally, let σ (resp. τ) be the (left) automorphism of $\mathbf{V}$ (resp. $\mathbf{E}$) induced by the coordinate permutation $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $\ldots$ $(2n-1) \leftrightarrow 2n$ (resp. $1 \leftrightarrow 2n$, $2 \leftrightarrow 3$, $\ldots$ $(2n-2) \leftrightarrow (2n-1)$). It is easily verified that

Theorem 7.1 *$(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ is a dihedral embedding.*

The dihedral embedding $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ defined above is called the *dihedral symmetrization of P*.

We come to the main result of this section.

Theorem 7.2 *Let P be an n-gon of groups, all of whose vertex angles are $\leq \pi/2$, and let S be the n-gon of the dihedral symmetrization of P, using the notation $(n, \mathbf{F}, \mathbf{E}, \mathbf{V}, \varphi, \psi, \sigma, \tau)$ as above. Then*

(i) The angle between $\sigma(\mathbf{E})$ and $\mathbf{E}$ in $\mathbf{V}$ with respect to $\mathbf{F}$ is $\pi/2$.

(ii) $\pi_1(P)$ *embeds in* $\pi_1(S)$*; in fact,* $\pi_1(P)$ *is isomorphic to a retract of* $\pi_1(S)$*.*

PROOF. To prove (i), we first recall that the vertex angle between two subgroups E and E_0 of a group V with respect to F is $\leq \pi/2$ if and only if $E \cap E_0 \leq F$. However,

$$\sigma(\mathbf{E}) \cap \mathbf{E} = (E_1, E_n, E_2, E_1, \ldots, E_n, E_{n-1}) \cap (E_n, E_1, E_1, E_2, \ldots, E_{n-1}, E_n) \tag{7.2}$$

$$= (E_1 \cap E_n, E_n \cap E_1, E_2 \cap E_1, E_1 \cap E_2, \ldots, E_n \cap E_{n-1}, E_{n-1} \cap E_n) \tag{7.3}$$

$$\leq (F, F, F, F, \ldots, F, F), \tag{7.4}$$

with the inclusion following because all of the vertex angles of P are $\leq \pi/2$. Therefore, the angle between $\sigma(\mathbf{E})$ and $\mathbf{E}$ in $\mathbf{V}$ with respect to $\mathbf{F}$ is $\leq \pi/2$.

Conversely, let e be an element of an edge group of P such that $e \notin F$. We may as well assume that $e \in E_1$. Then if $e_1 = (e, 1, 1, 1, \ldots, 1, 1)$ and $e_2 = (1, e, 1, 1, \ldots, 1, 1)$, $e_1 e_2 e_1^{-1} e_2^{-1}$ is a length 4 element of the kernel of the natural map from $\sigma(\mathbf{E}) *_{\mathbf{F}} \mathbf{E}$ to $\mathbf{V}$. It follows that the angle is exactly $\pi/2$.

As for (ii), the general case is again notationally and typographically quite complicated, so again we illustrate only the case of $n = 4$. Let P be the square in (7.1), and let S be its dihedral symmetrization. Consider injections from the face, edge, and vertex groups of P to the face, edge, and vertex groups of S, as indicated in Figure 7. A diagram chase shows that these injections embed P in S, and we have a naturally induced homomorphism $\Phi : \pi_1(P) \mapsto \pi_1(S)$. However, if Ψ is the homomorphism from $\pi_1(S)$ to $\pi_1(P)$ obtained by killing all of the coordinates in Figure 7 which have 1's in them, then $\Psi\Phi$ is the identity on $\pi_1(P)$, and (ii) follows.

Combining Theorem 7.2 with Theorem 6.3, we obtain the following result.

Corollary 7.3 *The fundamental group of any* n*-gon of groups with all of its vertex angles* $\leq \pi/2$ *can be embedded in the fundamental group of a triangle of groups with vertex angles* $\pi/2$*,* $\pi/4$*, and* π/n*.*

Since the face, edge, and vertex groups of the n-gon of the dihedral symmetrization of an n-gon are all finite direct products of the face, edge, and vertex groups of the original n-gon, Theorems 7.2 and 6.3 imply the Second Main Theorem (Theorem 1.2), which we restate here:

Corollary 7.4 *Let* R *be a square of finite groups, all of whose vertex angles are* $\leq \pi/2$*. Then* $\pi_1(R)$ *can be embedded in the fundamental group of a non-positively curved triangle of finite groups.*

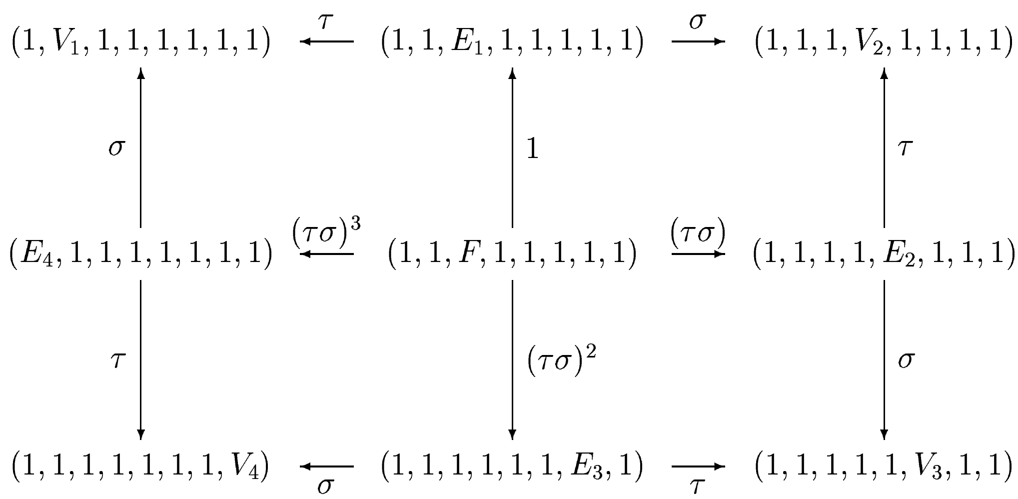

Fig. 7. Embedding of P in S used in the proof of Theorem 7.2

Remark 7.5 In fact, when R is a square of finite groups with all of its vertex angles $\leq \pi/2$, and with trivial face group, it is possible to embed $\pi_1(R)$ in a non-positively curved triangle of finite groups which has a vertex whose group is isomorphic to $Z_2 \times Z_2$, and whose adjoining edge subgroups are each isomorphic to Z_2, as follows. From Theorem 7.2, we know that R can be embedded in S, the square of the dihedral symmetrization of R. Furthermore, from Theorem 6.3, we know that if T is the triangle of the the dihedral symmetrization of R, then S is the eightfold cover of T corresponding to the quotient $\pi_1(T) \mapsto D_{2\cdot 4}$. However, if we instead consider the double cover of T corresponding to the quotient

$$\pi_1(T) \mapsto D_{2\cdot 4} \mapsto \langle t \rangle \cong Z_2, \tag{7.5}$$

where the map from $D_{2\cdot 4}$ to $\langle t \rangle$ is defined by killing s, we obtain a triangle T_2 of finite groups which has the desired vertex and edge groups. (The algebraically minded reader may verify the details of this last assertion by imitating the proof of part (ii) of Theorem 6.3.) The factorization (7.5) implies that $\pi_1(S)$ embeds in $\pi_1(T_2)$, which means that $\pi_1(R)$ embeds in $\pi_1(T_2)$ as well.

Finally, since $\pi/2 + \pi/4 + \pi/n < \pi$ if $n > 4$:

Corollary 7.6 *For $n > 4$, let R be an n-gon of finite groups with all of its vertex angles $\leq \pi/2$. Then $\pi_1(R)$ can be embedded in the fundamental group of a negatively curved triangle of finite groups.*

Remark 7.7 The most "natural" way of defining the dihedral symmetrization would use the free product instead of the direct product to construct $\mathbf{F}$, $\mathbf{E}$, and $\mathbf{V}$. In this variation, the proof of Theorem 7.2 goes through, with two added bonuses: the vertex angles of S are all equal to the maximum of the vertex angles of P; and as a group, $\pi_1(S)$ is precisely the free product of $2n$ copies of $\pi_1(P)$. The problem, of course, is that $\mathbf{F}$, $\mathbf{E}$, and $\mathbf{V}$ are no longer finite.

8 A conjecture

Finally, we wish to include the following conjecture, to give an idea of the richness of the set of groups which are subgroups of non-positively curved polygons of finite groups. Basically, we conjecture that if a cocompact group of isometries of a CAT(0) space *looks like* a subgroup of the fundamental group of a non-positively curved polygon of finite groups, then it *is* such a subgroup. We state the conjecture only for 2-dimensional CAT(0) actions, though we suspect that analogous statements hold for isometry groups of CAT(−1) complexes and negatively curved polygons of groups, as well as in higher dimensions. (It is easy to verify the conjecture in dimension 1).

Conjecture 8.1 *Let G be a group which acts combinatorially, properly discontinuously, and cocompactly by isometries on a piecewise Euclidean CAT(0) 2-complex X, let P be an n-sided Euclidean polygon, and let φ be a G-equivariant combinatorial map from X to P which restricts to an isometry on each cell of X. Then there exists a non-positively curved polygon of finite groups $R(P)$ over P, such that G may be identified with a subgroup of $\pi_1(R(P))$, and X may be identified equivariantly with a convex subcomplex of the universal cover $\widetilde{R(P)}$ in such a way that $\varphi : X \to P$ is the composition $X \hookrightarrow \widetilde{R(P)} \to P$.*

For instance, let Q be a compact non-positively curved squared complex, and suppose that there is an equivariant combinatorial map from $X = \tilde{Q}$ to the unit square P in the Euclidean plane which is an isometry on each cell of X. Conjecture 8.1 would then imply that there exists a right-angled square of finite groups $R(P)$ such that $\pi_1(Q)$ may be embedded in $\pi_1(R(P))$, and X may be identified equivariantly with a convex subcomplex of the universal cover $\widetilde{R(P)}$ of $R(P)$. Note that this result, used in place of the First Main Theorem, is enough to prove the square case of Corollary 1.4.

Similarly, another consequence of Conjecture 8.1 would be that if Q is a compact non-positively curved squared complex, then $\pi_1(Q)$ is a subgroup of the fundamental group of a right isoceles triangle of finite groups. In this case, the equivariant map from $X = \tilde{Q}$ to the right isoceles triangle is the obvious one corresponding to the subdivision of each square into eight right isoceles triangles (Figure 2). This result can be used in place of the First and Second Main Theorems to prove the triangle case of Corollary 1.4. In fact, the same equivariant map, applied to the case where X is the universal covering space of a square of finite groups R, and $G = \pi_1(R)$, yields the Second Main Theorem as a consquence.

One final consequence of Conjecture 8.1 uses the following result from [18].

Theorem 8.2 *There exists a non-virtually torsion-free group G which acts combinatorially, properly discontinuously, and cocompactly by isometries on a piecewise Euclidean CAT(0) squared 2-complex X. Furthermore, there is a G-equivariant combinatorial map from X to the unit Euclidean square which restricts to an isometry on each cell of X.*

If Conjecture 8.1 holds, Theorem 8.2 would yield examples of non-positively curved squares and triangles of finite groups with non-virtually torsion-free fundamental groups.

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