

GROUPS WITH INFINITELY MANY TYPES OF FIXED SUBGROUPS

TIM HSU AND DANIEL T. WISE

ABSTRACT. It is a theorem of Shor that if G is a word-hyperbolic group, then up to isomorphism, only finitely many groups appear as fixed subgroups of automorphisms of G . We give an example of a group G acting freely and cocompactly on a CAT(0) square complex such that infinitely many non-isomorphic groups appear as fixed subgroups of automorphisms of G . Consequently, Shor's finiteness result does not hold if the negative curvature condition is relaxed to either biautomaticity or nonpositive curvature.

1. INTRODUCTION

Let $\phi: G \rightarrow G$ be an automorphism of a group. The **fixed subgroup of ϕ** , denoted by $\text{Fix}(\phi)$, is defined as follows:

$$(1) \quad \text{Fix}(\phi) = \{g \in G: \phi(g) = g\}.$$

For example, if there exists $x \in G$ such that $\phi(g) = xgx^{-1}$ (i.e., if ϕ is inner), then $\text{Fix}(\phi) = C_G(x)$, the centralizer of x in G .

Finitely generated free groups have the remarkable property that their fixed subgroups are finitely generated. Specifically:

Theorem 1.1. *If $\phi: F_r \rightarrow F_r$ is an automorphism of a free group of rank r , then $\text{rank}(\text{Fix}(\phi)) \leq r$.*

Theorem 1.1 began as a conjecture attributed to P. Scott, and was initially proven in various geometric cases. Gersten [Ger87] was the first to prove that $\text{Fix}(\phi)$ is always finitely generated, and Bestvina and Handel [BH92] were the first to establish that $\text{rank}(\text{Fix}(\phi)) \leq r$.

Date: version of: September 7, 2004.

2000 Mathematics Subject Classification. 20F67, 20F28.

Key words and phrases. automorphisms, fixed subgroups, HNN extensions.

D. T. Wise was supported by grants from FCAR and NSERC..

See [Ven02] for a survey of this area, which has maintained continual activity.

Note that the finite rank hypothesis is crucial in Theorem 1.1. Indeed, for each n , the infinite rank free group F_∞ has an automorphism ϕ_n such that $\text{Fix}(\phi_n) \cong F_n$.

Another example worth noting is the group

$$(2) \quad F_2 \times \mathbb{Z} \cong \langle a, b, t \mid [a, t], [b, t] \rangle,$$

as the automorphism ϕ induced by $a \mapsto at, b \mapsto bt, t \mapsto t$ has $\text{Fix}(\phi) \cong F_\infty \times \mathbb{Z}$. Indeed, it is not hard to see that $\text{Fix}(\phi)$ is the subgroup of $F_2 \times \mathbb{Z}$ consisting of those elements with zero exponent sum in a and b . Nevertheless, $F_n \times \mathbb{Z}$ has only finitely many isomorphism types of fixed subgroups; more precisely, we leave it as an exercise to show that if ϕ is an automorphism of $F_n \times \mathbb{Z}$, then either $\phi(t) = t$, in which case $\text{Fix}(\phi)$ is isomorphic to $F_r \times \mathbb{Z}$ with $r \leq n$ or $r = \infty$, or $\phi(t) = t^{-1}$, in which case $\text{Fix}(\phi)$ is free of rank $r \leq 2n - 1$.

Recently, Shore [Sho99] gave the following interesting generalization of Theorem 1.1:

Theorem 1.2. *Let G be a word-hyperbolic group. Then up to isomorphism, only finitely many groups appear as fixed subgroups of automorphisms of G .*

In this paper, we give examples of finitely generated groups that have infinitely many non-isomorphic fixed subgroups. More precisely, we show that:

Theorem 1.3. *There exists a group G that acts freely and cocompactly on the cartesian product of two trees such that infinitely many non-isomorphic groups appear as fixed subgroups of automorphisms of G . In particular, there exist centralizers in G whose abelianizations are free abelian of arbitrarily high finite rank.*

We note that G is the fundamental group of a compact nonpositively curved square complex, which means that G is both a CAT(0) group and a $C(4)$ - $T(4)$ group, and thus biautomatic [GS91]. Therefore, Theorem 1.3 demonstrates a contrast between the behavior of word-hyperbolic groups and their semi-hyperbolic generalizations.

We begin the construction of G by reviewing **complete square complexes** in Section 2. In Sections 3 and 4, we describe the structure of centralizers in the fundamental group of a complete square complex, and in Section 5, we specialize that discussion to produce the desired example.

Notation 1.4. If H is a subgroup of G and $x \in G$, then the centralizer of x in H is denoted by $C_H(x)$. The length of a reduced word w is denoted by $|w|$. The abelianization G/G' of a group G is denoted by G_{ab} .

2. COMPLETE SQUARE COMPLEXES

Recall that a **square complex** is a combinatorial 2-complex whose 2-cells are squares. We are mainly interested in the following type of square complex, introduced in [Wis96].

Definition 2.1. A **complete square complex**, or **CSC**, is a square complex X such that:

1. The 1-cells of X are partitioned into two classes, X_V and X_H , that induce the structure of a complete bipartite graph on the link of any vertex of X . We think of X_V as the **vertical** 1-cells and X_H as the **horizontal** 1-cells.
2. The squares of X are oriented as shown in Figure 1, with vertical cells oriented “up” and horizontal cells oriented “right”, and the 1-cells of X are oriented in a compatible manner.

By a slight abuse of terminology, we will use X_V and X_H to refer to the graphs formed by taking the union of the appropriate 1-cells and the 0-cells of X . We also call the elements of $H = \pi_1(X_H)$ (resp. $V = \pi_1(X_V)$) corresponding to the oriented 1-cells of X_H (resp. X_V) the **standard generators** of H (resp. V).

For expository convenience, our notation and terminology differs somewhat from [Wis96], but is essentially equivalent. In particular, what we are calling a CSC here is actually a **directed \mathcal{VH} CSC**, in the terminology of [Wis96].

The fundamental groups of CSC’s have many interesting properties. For example, the universal cover of a CSC is the product of two trees [Wis96, Thm 1.10], so the fundamental group of a compact

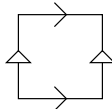


FIGURE 1. Orientation of the squares in a CSC

CSC acts freely and cocompactly on a CAT(0) space. We also have the following decomposition.

Definition 2.2. It is not hard to see that a CSC X has the following structure as a graph of spaces [SW79]:

1. The vertex spaces of X are the connected components of the graph X_H .
2. The edge spaces of X correspond to the connected components of $X - X_H$, and each edge space of X has the form $\Lambda \times [0, 1]$, where Λ is an oriented graph.
3. The ends $\Lambda \times 0$ and $\Lambda \times 1$ of an edge space $\Lambda \times [0, 1]$ are attached to their corresponding vertex spaces by covering maps.

We call this graph of spaces the **horizontal decomposition** of X . Note that if X_H is connected, and there is exactly one edge space in the horizontal decomposition of X , then we can give $\pi_1(X)$ the structure of an HNN extension of $H = \pi_1(X_H)$ by taking the stable letter to be any of the standard generators of $V = \pi_1(X_V)$.

Let X be a CSC, let $H = \pi_1(X_H)$, and let $V = \pi_1(X_V)$. We will need the following observations about $\pi_1(X)$, all of which can be deduced easily from the fact that \tilde{X} is isomorphic to the product $\tilde{X}_H \times \tilde{X}_V$ (see [Wis96, Thm 1.10]).

Lemma 2.3 (Normal forms). *The inclusion of X_H and X_V in X induces embeddings of H and V in $\pi_1(X)$ such that $H \cap V = 1$. Furthermore, if X has one 0-cell, then for any $\sigma \in \pi_1(X)$, there exist unique $h \in H$ and $v \in V$ such that $\sigma = vh$. \square*

Lemma 2.4 ($HV = VH$). *Suppose X has only one 0-cell, and let $h_0 \in H$ and $v_0 \in V$ be reduced words. Then for the unique reduced words $h_1 \in H$ and $v_1 \in V$ such that $h_0v_0 = v_1h_1$, we have $|h_1| = |h_0|$ and $|v_1| = |v_0|$. Furthermore, if h_0 (resp. v_0) is a positive standard generator of H (resp. V), then so is h_1 (resp. v_1). \square*

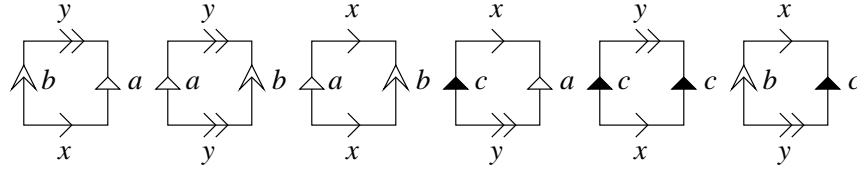


FIGURE 2. Example 2.5

Example 2.5. Let X be a square complex with one 0-cell, 1-cells $\{a, b, c, x, y\}$, and 2-cells defined by the six squares in Figure 2. Then X is a CSC with $X_V = \{a, b, c\}$ and $X_H = \{x, y\}$. The horizontal decomposition of X is illustrated in Figure 3. The bouquet of two circles on the right represents X_H , which is also the lone vertex space of the decomposition. The two graphs on the left are both combinatorially equivalent to a graph Λ , and the covering maps used to attach the edge space $\Lambda \times [0, 1]$ to X_H are given by the labels on the two graphs. Note that the vertices in the left-hand graphs are precisely the endpoints of a, b , and c , as indicated, and the vertical edges $\{a, b, c\}$ are oriented from the lower graph to the upper one.

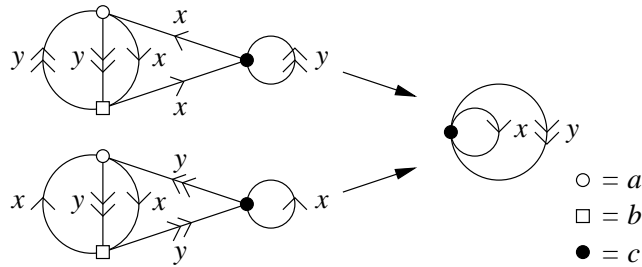


FIGURE 3. Horizontal decomposition of Example 2.5

3. HORIZONTAL CENTRALIZERS IN CSC GROUPS

Throughout this section, let X be a CSC with one vertex, let $G = \pi_1(X)$, let $H = \pi_1(X_H)$, let $V = \pi_1(X_V)$, and let c be a standard generator of V . We begin our study of centralizers in CSC groups by looking at $C_H(c^n)$, the “horizontal” part of the centralizer of c^n .

Let

$$(3) \quad H(m) = H \cap c^m H c^{-m},$$

and let $\tilde{X}_H(m)$ be the based cover of X_H corresponding to the subgroup $H(m) \leq H$. For example, when $m = 1$, $H(1)$ is one of the associated subgroups in the construction of G as an HNN extension of H , and $\tilde{X}_H(1)$ is precisely the graph Λ that appears in the horizontal decomposition of X (Definition 2.2).

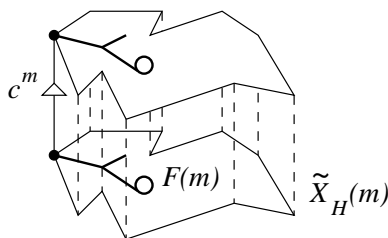


FIGURE 4. Parallel transporting $\tilde{X}_H(m)$ along c^m

We consider $H(m)$ because on the one hand, clearly $C_H(c^m) \leq H(m)$, and on the other hand, it turns out that $\tilde{X}_H(m)$ is the smallest cover of X_H that can be consistently “parallel transported” along the based path c^m , as sketched in Figure 4. (Figure 4 also gives an overview of the constructions in this section, including a subgraph $F(m) \subseteq \tilde{X}_H(m)$ that will be explained later.)

In the rest of this section, we explain what is meant by “parallel transporting” $\tilde{X}_H(m)$ along c^m . We begin by defining two useful functions.

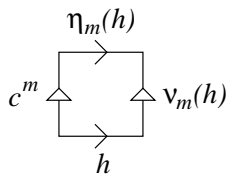


FIGURE 5. Definition of $\eta_m(h)$ and $\nu_m(h)$

Definition 3.1. By Lemma 2.4, for any $h \in H$, there exist unique $\eta_m(h) \in H$ and $\nu_m(h) \in V$ such that $h^{-1}c^m = \nu_m(h)\eta_m(h)^{-1}$. Therefore,

$$(4) \quad c^m \eta_m(h) = h \nu_m(h)$$

defines functions $\eta_m: H \rightarrow H$ and $\nu_m: H \rightarrow V$.

As illustrated in Figure 5, the main idea of Definition 3.1 is that $\eta_m(h)$ is supposed to be the “parallel transport” of the based path h along c^m .

The first goal of this section is to define the **cylinder of $H(m)$** in Definition 3.5. We begin by showing in Lemmas 3.3 and 3.4 that the labels that we will use on the cylinder of $H(m)$ are well-defined.

Notation 3.2. In the sequel, we will freely identify any $h \in H$ with the unique based path it defines in $\tilde{X}_H(m)$.

Lemma 3.3. For $h \in H(m)$, $\nu_m(h) = c^m$. More generally, for any based path $h \in H$, the value of $\nu_m(h)$ depends only on the endpoint of h in $\tilde{X}_H(m)$.

Proof. Suppose $h \in H(m)$. Since $h = c^m h' c^{-m}$ for some $h' \in H$, by Equation (4), we have

$$(5) \quad c^m \eta_m(h) = h \nu_m(h) = c^m h' c^{-m} \nu_m(h),$$

which means that $\eta_m(h) = h'(c^{-m} \nu_m(h))$. Therefore, since $\eta_m(h)$ and h' are in H and $c^{-m} \nu_m(h)$ is in V , the uniqueness of normal forms (Lemma 2.3) implies that $c^{-m} \nu_m(h) = 1$. The first assertion follows.

As for the second assertion, suppose h_1 and h_2 have the same endpoint in $\tilde{X}_H(m)$. Then by Equation (4),

$$(6) \quad \begin{aligned} \nu_m(h_2)^{-1} \nu_m(h_1) &= (h_2^{-1} c^m \eta_m(h_2))^{-1} (h_1^{-1} c^m \eta_m(h_1)) \\ &= \eta_m(h_2)^{-1} c^{-m} h_2 h_1^{-1} c^m \eta_m(h_1). \end{aligned}$$

However, since $h_2 h_1^{-1} \in H(m)$, the first assertion of the lemma implies that $\nu_m(h_2 h_1^{-1}) = c^m$, which means that $c^{-m} (h_2 h_1^{-1}) c^m = \eta_m(h_2 h_1^{-1})$. Therefore,

$$(7) \quad \nu_m(h_2)^{-1} \nu_m(h_1) = \eta_m(h_2)^{-1} \eta_m(h_2 h_1^{-1}) \eta_m(h_1) \in H \cap V = 1,$$

and the second assertion follows. \square

Lemma 3.4. *For any $h \in H$ and any standard generator e of H , $\eta_m(h)^{-1}\eta_m(he)$ is also a standard generator of H , depending only on e and the endpoint of h .*

Proof. By Equation (4), we have $\nu_m(h) = h^{-1}c^m\eta_m(h)$ and $\nu_m(he) = (he)^{-1}c^m\eta_m(he)$, so

$$(8) \quad \nu_m(h)^{-1}e\nu_m(he) = \eta_m(h)^{-1}\eta_m(he).$$

Then, on the one hand, the left-hand side of Equation (8) depends only on e and the endpoint of h (Lemma 3.3); and on the other hand, since

$$(9) \quad e\nu_m(he) = \nu_m(h)(\eta_m(h)^{-1}\eta_m(he)),$$

by Lemma 2.4, $\eta_m(h)^{-1}\eta_m(he)$ is a standard generator of H . \square

We may now define our main construction, which gives a precise meaning to the “parallel transport” sketch in Figure 4.

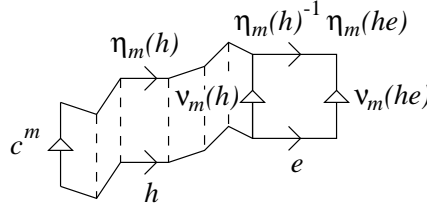


FIGURE 6. Labelling the cylinder of $H(m)$

Definition 3.5. The **cylinder of $H(m)$** is the (labelled) square complex $C(m)$ constructed as follows.

1. Topologically, let $C(m)$ be $\tilde{X}_H(m) \times [0, 1]$ (see Figure 4).
2. Label $\tilde{X}_H(m) \times 0$ with the corresponding labels of $\tilde{X}_H(m)$. (In fact, since $\tilde{X}_H(m) \times 0$ is labelled like $\tilde{X}_H(m)$, we will often treat them as identical in the sequel.)
3. For any based path $h \in H$, label the “vertical” edge starting at the endpoint of h in $\tilde{X}_H(m) \times 0$ with the element $\nu_m(h)$, as shown in Figure 6. Note that by Lemma 3.3, this depends only on the endpoint of h . Furthermore, since $\tilde{X}_H(m)$ is connected, this rule labels all vertical edges in $C(m)$.

4. For any based path $h \in H$ in $\tilde{X}_H(m) \times 0$, and any standard generator e of H , label the edge of $\tilde{X}_H(m) \times 1$ that lies above the last edge of the path he with the standard generator $\eta_m(h)^{-1}\eta_m(he)$ of H (see Lemma 3.4), as shown on the right-hand side of Figure 6.

For example, $C(1)$ is precisely the edge space of the horizontal decomposition of X , and the 1-cells of $C(1)$ are labelled as the corresponding 1-cells of that edge space.

Definition 3.6. In the notation of Definition 3.5, we define the **parallel transport** of any edge (resp. path) in $\tilde{X}_H(m) \times 0$ to be the corresponding edge (resp. path) in $\tilde{X}_H(m) \times 1$.

We now come to the key property of $C(m)$.

Lemma 3.7. *If $h \in H$ is a based path in $\tilde{X}_H(m) \times 0$, then the parallel transport of h is the based path $\eta_m(h)$ in $\tilde{X}_H(m) \times 1$.*

Proof. This follows from an easy induction on the length of h , as indicated by Figure 6. □

Definition 3.8. Let $F_0(m)$ be the union of the basepoint of $\tilde{X}_H(m) \times 0$ and all edges in $\tilde{X}_H(m) \times 0$ whose parallel transports in $\tilde{X}_H(m) \times 1$ are labelled with the same generator. We define the **fixed graph** $F(m)$ to be the connected component of $F_0(m)$ that contains the basepoint of $\tilde{X}_H(m) \times 0$.

The idea of $F(m)$ is illustrated in Figure 4. The following lemma gives two alternate descriptions of $F(m)$.

Lemma 3.9. *The fixed graph $F(m)$ is precisely the union of all based paths h in $\tilde{X}_H(m) \times 0$ such that $\eta_m(h) = h$. In other words, a based path h is contained in $F(m)$ if and only if $h^{-1}c^mh \in V$.*

Proof. The first statement follows immediately from Lemma 3.7. The second statement follows from the first statement and the definition of η_m (Definition 3.1). □

We now come to our main tool for analyzing the centralizer of c^m .

Theorem 3.10. *The inclusion of the fixed graph $F(m)$ in $\tilde{X}_H(m)$ induces an isomorphism from $\pi_1(F(m))$ onto $C_H(c^m) \leq H(m) = \pi_1(\tilde{X}_H(m))$.*

Proof. On the one hand, let h be a based loop in $F(m)$. Since $h \in H(m)$, we have $\nu_m(h) = c^m$ (Lemma 3.3), and since h is in $F(m)$, we have $\eta_m(h) = h$ (Lemma 3.9). Therefore, by Equation (4), $c^m h = hc^m$, which means that $h \in C_H(c^m)$.

Conversely, let h be a reduced element of $C_H(c^m)$. Since h is also in $H(m)$, h defines a based loop in $\tilde{X}_H(m)$ and $\eta_m(h) = c^m h c^{-m}$ (Lemma 3.3 and Equation (4)). However, if the path h ever leaves $F(m)$, then h has an edge that changes in its parallel transport, which means that $c^m h c^{-m} = \eta_m(h) \neq h$ (since $\eta_m(h)$ is also a reduced path). Therefore, h must be contained in $F(m)$. The theorem follows. \square

Our last theorem of this section (Theorem 3.12) describes how the different subgraphs $F(m)$ fit together. We first need the following lemma.

Lemma 3.11. *For $n > m$, we have $H(n) \leq H(m)$, which means that $\tilde{X}_H(n)$ is a based cover of $\tilde{X}_H(m)$.*

Proof. Thinking of G as an HNN extension of H with stable letter c , we note that for $h \in H$, $h \in H(n)$ if and only if $c^{-n} h c^n$ reduces to some $h' \in H$. However, by the normal form theorem for HNN extensions, for $0 < m < n$, if $c^{-m} h c^m$ cannot be reduced to an element of H , then $c^{-n} h c^n$ also cannot be reduced to an element of H . In other words, any element of $H(n)$ must also be an element of $H(m)$. \square

Theorem 3.12. *For $m \geq 1$, $F(m)$ is a subgraph of $F(2m)$. In particular, $F(1) \subseteq F(2) \subseteq \dots \subseteq F(2^k) \subseteq \dots$ is an ascending chain of graphs.*

Proof. Consider the cover $\tilde{X}_H(2m) \rightarrow \tilde{X}_H(m)$ from Lemma 3.11. The inclusion of $F(m)$ in $\tilde{X}_H(m)$ lifts to a map from $F(m)$ into $\tilde{X}_H(2m)$ if and only if every element of $\pi_1(F(m))$ lifts to a loop in $\pi_1(\tilde{X}_H(2m))$. Therefore, since Theorem 3.10 implies

$$(10) \quad \pi_1(F(m)) = C_H(c^m) \leq C_H(c^{2m}) \leq H(2m) = \pi_1(\tilde{X}_H(2m)),$$

we may consider $F(m)$ as a based subgraph of $\tilde{X}_H(2m)$.

By Lemma 3.9, it now suffices to show that every based path h in $F(m)$ is also contained in $F(2m)$. So suppose h is contained in $F(m)$. In that case, $h^{-1}c^mh \in V$ (Lemma 3.9), which means that $(h^{-1}c^mh)^2 = h^{-1}c^{2m}h \in V$. Therefore, h is contained in $F(2m)$ (Lemma 3.9 again). The theorem follows. \square

4. FULL CENTRALIZERS IN CSC GROUPS

In this section, retaining the notation and conventions of Section 3, we extend our analysis of $C_H(c^m)$ to the full centralizer $C_G(c^m)$. We begin by looking at how c acts on $C_H(c^m)$.

Lemma 4.1. *For $m \geq 1$, there exists a graph automorphism γ_m of $F(m)$ such that:*

1. *The automorphism of $C_H(c^m) \cong \pi_1(F(m))$ induced by γ_m is precisely conjugation by c . (In particular, c normalizes $C_H(c^m)$.)*
2. *Considering $F(m)$ as a subgraph of $F(2m)$ (see Theorem 3.12), the restriction of γ_{2m} to $F(m)$ is precisely γ_m .*

Proof. First, following Definition 3.1, by Lemma 2.4, we may define functions $\phi: H \rightarrow H$ and $\psi: H \rightarrow V$ by the equation

$$(11) \quad c\phi(h) = h\psi(h).$$

Then, since $H(m) \leq H(1)$, the proof of Lemma 3.3 shows that $\phi(h)$ depends only on the endpoint of a based path h in $\tilde{X}_H(m)$, and the proof of Lemma 3.4 shows that ϕ defines a relabelling of $\tilde{X}_H(m)$ analogous to the one in Definition 3.5 and Lemma 3.7.

Next, consider any based path h contained in the fixed graph $F(m)$. By Lemma 3.9, we have that $\eta_m(h) = h$, which means that Equation (4) implies $h^{-1}c^mh = \nu_m(h)$. Applying Equation (11), we then see that

$$(12) \quad \phi(h)^{-1}c^m\phi(h) = \psi(h)^{-1}h^{-1}c^mh\psi(h) = \psi(h)^{-1}\nu_m(h)\psi(h) \in V,$$

which means that $\phi(h)$ is contained in $F(m)$ (Lemma 3.9 again). Therefore, the relabelling defined by ϕ preserves $F(m)$ setwise. Consequently, since $F(m)$ is a finite graph, ϕ must actually induce an automorphism γ_m of $F(m)$. Assertion 2 of the lemma follows immediately, since the automorphisms γ_m are induced by the relabelling ϕ , which is independent of m .

As for the rest of the lemma, suppose $h \in \pi_1(F(m))$. First, Lemma 2.4 and Equation (11) imply that $\psi(h)$ is a standard generator of V . Furthermore, since $h \in \pi_1(F(m)) \leq H(m)$, we see that $\nu_m(h) = c^m$, and Equation (12) implies

$$(13) \quad c^m \phi(h) = \phi(h)(\psi(h)^{-1} c^m \psi(h)).$$

Applying Lemma 2.4 again, we see that $|\psi(h)^{-1} c^m \psi(h)| = |c^m|$. Therefore, since $C_V(c^m) = \langle c \rangle$, we must have $\psi(h) = c$ and $\phi(h) = c^{-1} h c$. The lemma follows. \square

Remark 4.2. Note that assertion 2 of Lemma 4.1 is equivalent to saying that we may combine the automorphisms γ_{2^k} to define an automorphism γ of the direct limit $F(1) \subseteq F(2) \subseteq \cdots \subseteq F(2^k) \subseteq \cdots$. We will refer to γ freely in the sequel.

Lemma 4.3. *For $m \geq 1$, $C_G(c^m) = C_H(c^m) \rtimes \langle c \rangle$, where the semidirect product is defined by the action in Lemma 4.1.*

Proof. By Lemma 4.1, it suffices to show that $C_G(c^m) = \langle c \rangle C_H(c^m)$. So consider $g \in C_G(c^m)$. By the Normal Form Lemma, we have $g = g_v g_h$, where g_v (resp. g_h) is a reduced word in V (resp. H). It will therefore be enough to show that if the first letter of g_v is not $c^{\pm 1}$, then $g_v = 1$.

So suppose that the first letter of g_v is not $c^{\pm 1}$. By Lemma 2.4, $g_h c^m = k_v k_h$ for some reduced word $k_h \in H$ and some reduced word $k_v \in V$ such that $|k_v| = |c^m| = m$. But then, since $g_v g_h \in C_G(c^m)$, it follows that

$$(14) \quad c^m g_v g_h = g_v g_h c^m = g_v k_v k_h.$$

Therefore, by the uniqueness of normal forms (Lemma 2.3), we have $c^m g_v = g_v k_v$, or in other words, $k_v = g_v^{-1} c^m g_v$. However, since $g_v^{-1} c^m g_v$ is reduced, and $|k_v| = m$, we must have $g_v = 1$. \square

Theorem 4.4. *Let $\Gamma = \langle \gamma \rangle$. We have that $(C_G(c^m))_{\text{ab}} \cong \langle c \rangle \times \pi_1(\Gamma \backslash F(m))_{\text{ab}}$. Consequently, if the rank of $\pi_1(F(2^k))$ is a strictly increasing function of k , then the groups $(C_G(c^m))_{\text{ab}}$ are free abelian of arbitrarily high rank.*

Proof. On the one hand, since γ acts by conjugation on $\pi_1(F(m))$, the image of $\pi_1(F(m))$ in $(C_G(c^m))_{\text{ab}}$ must certainly factor through $\pi_1(\Gamma \backslash F(m))_{\text{ab}}$. On the other hand, γ has trivial action on $\Gamma \backslash F(m)$,

so c commutes with elements of $\pi_1(\Gamma \backslash F(m))$. The first claim of the theorem follows.

As for the other claim, if the rank of $\pi_1(F(2^k))$ is a strictly increasing function of k , since the action of γ_m commutes with the direct limit $F(1) \subset F(2) \subset \dots \subset F(2^k) \subset \dots$, there will be new topology at every stage of the direct limit $\Gamma \backslash F(1) \subset \Gamma \backslash F(2) \subset \dots \subset \Gamma \backslash F(2^k) \subset \dots$. Therefore, the rank of $\pi_1(\Gamma \backslash F(2^k))$ is a strictly increasing function of k , and the theorem follows. \square

5. THE EXAMPLES X AND X^+

We now specialize the results of Sections 3 and 4 to two particular examples. We again retain the notation and conventions of Sections 3 and 4 in this section.

Consider the CSC X from Example 2.5. Since X has one 0-cell, the results of Sections 3 and 4 apply to X and the element $c \in \pi_1(X)$. Furthermore, we deduce from [Wis96] that X has the following remarkable property.

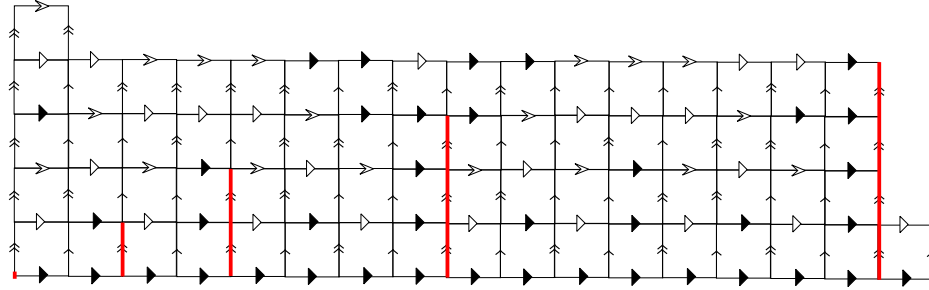


FIGURE 7. Positive quadrant of the antitorus A

Theorem 5.1. *The fixed graphs $F(2^k)$ form a strictly increasing sequence $F(1) \subset F(2) \subset \dots \subset F(2^k) \subset \dots$, with new edges and vertices at each stage.*

Proof. By Theorem 3.12, we need only show that there are always new edges and vertices in the fixed graph each time we go from $F(2^k)$ to $F(2^{k+1})$, and this follows from the construction of the **antitorus** A in [Wis96, Section II.3]. Briefly, A is a flat plane in \tilde{X} whose axes are labelled by infinite paths of the form c^∞ and y^∞ . The positive

quadrant of A is illustrated in Figure 7 (reflected along the diagonal to conserve space), using the notation of Example 2.5 and Figure 2. For our purposes, the key feature of A is the fact that it is aperiodic; specifically, the period of each successive infinite horizontal strip is double the period of the previous strip. In particular, the theorem follows from the fact that for each $k \geq 0$, the path y^{k+1} lies in $F(2^{k+1})$ but not in $F(2^k)$, as indicated by the heavy lines in Figure 7. \square

By Theorem 4.4, it then remains to show that the sequence $F(1) \subset F(2) \subset \dots \subset F(2^k) \subset \dots$ has new topology at each stage. This probably holds for the example X , but for the sake of brevity, we instead turn to a modified version of X .

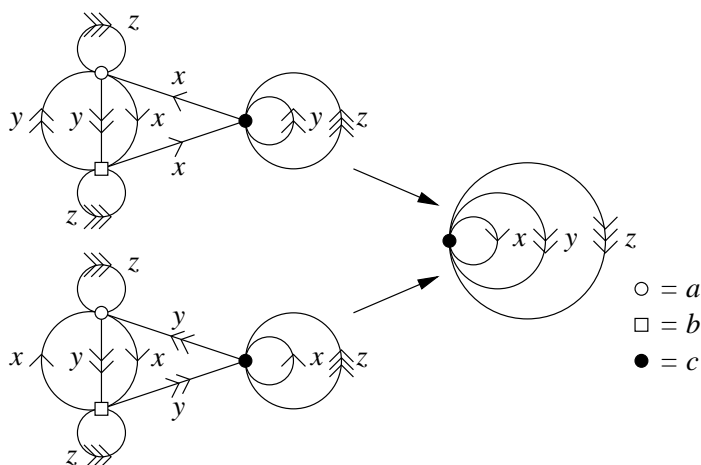


FIGURE 8. Horizontal decomposition of X^+

Proof of Theorem 1.3. Let X^+ be the CSC with one 0-cell, 1-cells $X_V^+ = \{a, b, c\}$ and $X_H^+ = \{x, y, z\}$, and the horizontal decomposition shown in Figure 8. Note that X^+ is just the example X plus an extra horizontal generator z that centralizes V . Let $G^+, H^+, H^+(m), \tilde{X}_H^+(m), \eta_m^+, \nu_m^+$, and $F^+(m)$ be the constructions for X^+ analogous to $G, H, H(m), \tilde{X}_H(m), \eta_m, \nu_m$, and $F(m)$ for X .

We first claim that each graph $\tilde{X}_H^+(m)$ is precisely the graph $\tilde{X}_H(m)$ with a z self-loop added at every vertex. This is equivalent to saying that for every $h \in H^+$, $hzh^{-1} \in H^+(m)$, which follows because

Equation (4) and the fact that z centralizes V imply

$$\begin{aligned}
 (15) \quad c^{-m}(hzh^{-1})c^m &= \eta_m^+(h)\nu_m^+(h)^{-1}z\nu_m^+(h)\eta_m^+(h)^{-1} \\
 &= \eta_m^+(h)z\eta_m^+(h)^{-1} \in H^+,
 \end{aligned}$$

as illustrated in Figure 9.

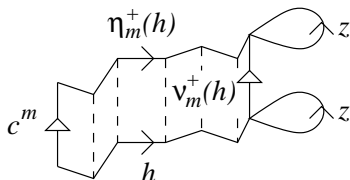


FIGURE 9. All z self-loops are fixed in parallel transport

We now observe that if h is contained in $F^+(m)$, then $\eta_m^+(h) = h$, and Equation (15) becomes $hzh^{-1}c^m = c^mhzh^{-1}$. It follows that every z self-loop at a vertex of $F^+(m)$ is included in $F^+(m)$, which means that $F^+(m)$ is precisely the graph $F(m)$ with a z self-loop added at every vertex. Therefore, from Theorem 5.1, we see that the strictly increasing sequence $F^+(1) \subset F^+(2) \subset \dots \subset F^+(2^k) \subset \dots$ has new topology at each stage. Theorem 4.4 then implies that the groups $(C_{G^+}(c^m))_{\text{ab}}$ are free abelian of arbitrarily high finite rank, and Theorem 1.3 follows. \square

REFERENCES

- [BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.
- [Ger87] S. M. Gersten. Fixed points of automorphisms of free groups. *Adv. in Math.*, 64(1):51–85, 1987.
- [GS91] S. M. Gersten and H. Short. Small cancellation theory and automatic groups. II. *Invent. Math.*, 105(3):641–662, 1991.
- [Sho99] Jeremy Andrew Shor. *On fixed subgroups of automorphisms in hyperbolic groups*. PhD thesis, Columbia University, 1999.
- [SW79] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.
- [Ven02] E. Ventura. Fixed subgroups in free groups: a survey. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*,

volume 296 of *Contemp. Math.*, pages 231–255. Amer. Math. Soc., Providence, RI, 2002.

- [Wis96] Daniel T. Wise. *Non-positively curved squared complexes, aperiodic tilings, and non-residually finite groups*. PhD thesis, Princeton University, 1996.

DEPT. OF MATH., SAN JOSE STATE UNIV., SAN JOSE, CA 95192-0103,
USA

E-mail address: hsu@math.sjsu.edu

DEPT. OF MATH. & STATS., MCGILL UNIV., MONTREAL, QUEBEC, H3A
2K6, CANADA

E-mail address: wise@math.mcgill.ca