Article Submitted to European Journal of Combinatorics

Partitioning the Boolean lattice into a minimal number of chains of relatively uniform size

Tim Hsu[1], Mark J. Logan[2], Shahriar Shahriari[3] and Christopher Towse[4]

[1]*Department of Mathematics, San José State University, San José, CA 95192-0103*

[2]*Department of Mathematics, University of Minnesota, Morris, Morris, MN 56267*

[3]*Department of Mathematics, Pomona College, Claremont, CA 91711*

[4]*Department of Mathematics, Scripps College, Claremont, CA 91711*

Abstract

Let $\mathbf{2}^{[n]}$ denote the *Boolean lattice* of order n, that is, the poset of subsets of $\{1, \ldots, n\}$ ordered by inclusion. Extending our previous work on a question of Füredi, we show that for any $c > 1$, there exist functions $e(n) \sim \sqrt{n}/2$ and $f(n) \sim c\sqrt{n \log n}$ and an integer N (depending only on c) such that for all $n > N$, there is a chain decomposition of the Boolean lattice $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, all of which have size between $e(n)$ and $f(n)$. (A positive answer to Füredi's question would imply that the same result holds for some $e(n) \sim \sqrt{\pi/2}\sqrt{n}$ and $f(n) = e(n)+1$.) The main tool used is an apparently new observation about *rank-collection* in normalized matching (LYM) posets.

1. Introduction

Let $[n] = \{1, \ldots, n\}$ and let $\mathbf{2}^{[n]}$ denote the Boolean lattice of order n, that is, the poset of subsets of $[n]$ ordered by inclusion. A collection of subsets $A_0 \subset \cdots \subset A_k$ of $[n]$ is called a chain of size $k + 1$ (or length k) in $\mathbf{2}^{[n]}$. In this paper, we show:

Main Theorem. *For any $c > 1$, there exist functions $e(n) \sim \frac{1}{2}\sqrt{n}$ and $f(n) \sim c\sqrt{n \log n}$ and an integer N (depending only on c) such that for all $n > N$, there is a decomposition of the Boolean lattice $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, all of which have size between $e(n)$ and $f(n)$.*

This paper is motivated by a question of Füredi [4], who asked if $\mathbf{2}^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains (the minimum number of chains in any chain decomposition of $\mathbf{2}^{[n]}$) such that the size of every chain is one of two consecutive integers. (This question was later generalized as a conjecture of Griggs [5]; see [7] for more background, context, and references.) If such a partition exists, by Stirling's formula, each chain in the partition must have size $a(n)$ or $a(n)+1$, for some $a(n) \sim \sqrt{\pi/2}\sqrt{n}$. In [7], we showed that $\mathbf{2}^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains such that the size of every chain is at least $e(n)$, where $e(n) \sim \frac{1}{2}\sqrt{n}$.

Thinking of our previous work as a partial answer to the lower bound part of Füredi's question, in this paper, we obtain a partial answer to the upper bound part of Füredi's question. Our main tool, which we call the *rank-collection theorem* (Theorem 2.2), is a matching property of normalized matching posets that is not hard to prove, but does not seem to have been widely used before. More specifically, we prove the Main Theorem by applying the rank-collection theorem, some facts on log-concavity (Section 3), and some estimates of binomial coefficients (Section 4), in a manner described in Section 5. (In fact, the reader may wish to begin with Section 5, to get a better idea of our overall strategy.)

Throughout this paper, we use $f(n) \sim g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = 1$, and $O(g(n))$ to denote a function $f(n)$ such that $\lim_{n\to\infty} f(n)/g(n) < \infty$.

2. Normalized matching

In this section, we summarize some facts about normalized matching, and obtain our main tool (Theorem 2.2).

Definition 2.1. Let P be a ranked poset. We say that P is *normalized matching* if, for any levels X and Y in P and $Z \subseteq X$, we have

$$\frac{|\Gamma(Z)|}{|Y|} \geq \frac{|Z|}{|X|}, \tag{1}$$

where $\Gamma(Z)$ is the set of neighbors of Z in Y.

For example, the Boolean lattice $\mathbf{2}^{[n]}$ is normalized matching. We also note that the normalized matching property is well-known to be equivalent to the *LYM property* of a ranked poset. (See Anderson [1, Ch. 2] for more on the normalized matching and LYM properties.)

The following theorem is not hard to prove, but does not seem to have been widely used before.

Theorem 2.2 (Rank-collection). *Let P be a normalized matching poset, and let Y_1 and Y_2 be consecutive levels of P. If P' is the poset obtained from P by combining Y_1 and Y_2 in a single level Y and ignoring all relations between elements of Y_1 and Y_2, then P' is also normalized matching.*

Proof. First, since (1) works "locally" (level by level), we need only verify (1) for P' when Y is one of the levels involved. In fact, it is enough to verify (1) for X a level of P' not equal to Y and $Z \subseteq X$, as the analogous inequality for subsets $W \subseteq Y$ follows by considering complements (see the proof of [3, Prop. 4.5.2]).

So let X be a level of P' not equal to Y, let Z be a subset of X, let $\Gamma(Z)$ be the set of neighbors of Z in level Y of P', and for $i = 1, 2$, let $\Gamma_i(Z)$ be the set of neighbors of Z in level Y_i of P. Since P is normalized matching, for $i = 1, 2$, we have:

$$|\Gamma_i(Z)| \geq |Z| \left(\frac{|Y_i|}{|X|} \right). \tag{2}$$

Therefore, since Y_1 and Y_2 are disjoint,

$$|\Gamma(Z)| = |\Gamma_1(Z)| + |\Gamma_2(Z)| \geq |Z| \left(\frac{|Y_1| + |Y_2|}{|X|} \right) = |Z| \left(\frac{|Y|}{|X|} \right). \tag{3}$$

The theorem follows. □

Corollary 2.3. *Let P be a normalized matching poset, let X, Y_1, and Y_2 be levels of P such that X is either above both or below both Y_1 and Y_2, and suppose that $|X| \geq |Y_1| + |Y_2|$. Then there exists a matching from $Y_1 \cup Y_2$ to a subset of X.*

Proof. Form a new poset P' by removing the levels of P between Y_1 and Y_2, and then combining the old levels Y_1 and Y_2. Since any rank-selected poset of a normalized matching poset is still normalized matching, the rank-collection theorem implies that P' is normalized matching. Since there exists a matching from the smaller of any two levels of a normalized matching poset to the larger (see Griggs [6]), the corollary follows. □

Remark 2.4. More generally, if we define a *rank-collected* poset of a given poset P to be any poset P' obtained from P by repeatedly removing levels and combining consecutive levels, then Theorem 2.2 implies that any rank-collected poset of a normalized matching poset is normalized matching.

3. Log-concavity

In this section, we summarize some facts about log-concavity.

Definition 3.1. Let $\ell \leq m$ be integers. A sequence $\{a_k\}$ of positive numbers is said to be *log-concave* for $\ell \leq k \leq m$ if

$$a_k^2 \geq a_{k+1} a_{k-1} \tag{4}$$

for all k such that $\ell < k < m$, or equivalently, if

$$\frac{a_k}{a_{k+1}} \geq \frac{a_{k-1}}{a_k} \tag{5}$$

for all k such that $\ell < k < m$.

For example, recall that for fixed n, the sequence $a_k = \binom{n}{k}$ is log-concave for $0 \leq k \leq n$. By the case $j = 0$ of [7, Cor. 4.12], we also see that:

Lemma 3.2. *The sequence* $b_k = \binom{n}{k} - \binom{n}{k-1}$ *is log-concave for* $0 \leq k \leq \lfloor n/2 \rfloor$. □

We will also need the following lemmas.

Lemma 3.3. *For any integers* a *and* m *such that* $0 < m < a$*, we have*

$$(a(a-m))^{(m+1)/2} \leq \prod_{i=0}^{m}(a-i) \leq \left(\left(\frac{2a-m}{2}\right)^2\right)^{(m+1)/2}. \tag{6}$$

Proof. Since any positive arithmetic sequence is log-concave,

$$a(a-m) \leq (a-1)(a-m+1) \leq \cdots \leq \left(\frac{2a-m}{2}\right)^2. \quad \square \tag{7}$$

Lemma 3.4. *Let* m *be a nonnegative integer, and let* $\{a_k\}$ *be a log-concave sequence such the sequence* $b_k = a_k - a_{k-1}$ *is also log-concave (and therefore, positive) for* $0 \leq k \leq m$*. If* $b_\lambda \geq a_\ell$ *for some* $\ell \leq \lambda \leq m$*, then* $b_{\lambda-j} \geq a_{\ell-j}$ *for* $1 \leq j \leq \ell$*.*

Note that by Lemma 3.2, the sequences $a_k = \binom{n}{k}$ and $b_k = \binom{n}{k} - \binom{n}{k-1}$ satisfy the first conditions of Lemma 3.4, taking $m = \lfloor n/2 \rfloor$.

Proof. Note that it is enough to show that $b_\lambda \geq a_\ell$ implies $b_{\lambda-1} \geq a_{\ell-1}$, as the general case follows by an easy induction. In fact, it is enough to show that

$$\frac{b_{\lambda-1}}{b_\lambda} \geq \frac{a_{\ell-1}}{a_\ell}, \tag{8}$$

as we may then multiply $b_\lambda \geq a_\ell$ by (8) to obtain the desired inequality.

Now, by the log-concavity of $\{a_k\}$ and (4), we see that

$$a_\ell a_{\ell-1} - a_\ell a_{\ell-2} \geq a_\ell a_{\ell-1} - a_{\ell-1}^2. \tag{9}$$

Dividing both sides by $a_\ell b_\ell = a_\ell(a_\ell - a_{\ell-1}) > 0$, we next see that

$$\frac{b_{\ell-1}}{b_\ell} \geq \frac{a_{\ell-1}}{a_\ell}. \tag{10}$$

However, since $\{b_k\}$ is log-concave, and since $\lambda \geq \ell$, repeated application of (5) to the sequence $\{b_k\}$ then yields (8). The lemma follows. □

4. Gaps and levels in $\mathbf{2}^{[n]}$

In this section, we give an estimate comparing the sizes of the "gaps" $\binom{n}{\lambda} - \binom{n}{\lambda-1}$ between consecutive levels of $\mathbf{2}^{[n]}$ and the level sizes $\binom{n}{k}$ themselves.

Throughout this section, we fix a constant $c > 1$, and we let

$$A(n) = c\sqrt{\log n}. \tag{11}$$

We begin with the following lemma, whose purpose will become clear shortly.

Lemma 4.1. *Let $\xi(n)$ be a function such that $\xi(n) \sim 2A(n)n^{-1/2}$. Then for sufficiently large n,*

$$(1+\xi(n))^{\frac{1}{4}(A(n)-1)\sqrt{n+2}} \geq \sqrt{n}. \tag{12}$$

Proof. Taking the log of both sides of (12), we see that it is enough to show that

$$\frac{\sqrt{n+2}(A(n)-1)\log(1+\xi(n))}{4} \geq \frac{1}{2}\log n \tag{13}$$

for sufficiently large n.

Since $\lim_{n\to\infty} \xi(n) = 0$, the Taylor expansion of $\log(1+x)$ shows that

$$\begin{aligned} \frac{\sqrt{n+2}(A(n)-1)\log(1+\xi(n))}{4} &\geq \frac{\sqrt{n+2}(A(n)-1)(\xi(n)-\xi(n)^2/2)}{4} \\ &= \frac{\sqrt{n+2}(A(n)-1)\xi(n)}{4} \cdot \left(1 - \frac{\xi(n)}{2}\right) \end{aligned} \tag{14}$$

for sufficiently large n.

Now, since $\sqrt{n+2}(A(n)-1)\xi(n) \sim 2(A(n))^2$, for sufficiently large n, we see that

$$\begin{aligned} \frac{\sqrt{n+2}(A(n)-1)\log(1+\xi(n))}{4} &\geq \frac{A(n)^2}{2} \cdot \left(1 - \frac{\xi(n)}{2}\right)(1+\epsilon(n)) \\ &= \frac{c^2 \log n}{2} \cdot \left(1 - \frac{\xi(n)}{2}\right)(1+\epsilon(n)), \end{aligned} \tag{15}$$

where $\lim_{n\to\infty} \epsilon(n) = 0$. Therefore, since $c^2 > 1$ and $\lim_{n\to\infty} \xi(n) = 0$, (13) holds for sufficiently large n, and the lemma follows. ◻

For $n \geq 0$, let

$$\lambda(n) = \left\lceil \frac{n}{2} - \frac{\sqrt{n+2}}{2} \right\rceil, \qquad k(n) = \left\lfloor \frac{n}{2} - \frac{A(n)\sqrt{n+2}}{2} \right\rfloor. \tag{16}$$

Theorem 4.2. *If $\lambda = \lambda(n)$ and $k = k(n)$ are defined by (16), then*

$$\binom{n}{\lambda} - \binom{n}{\lambda - 1} \geq \binom{n}{k} \tag{17}$$

for sufficiently large n.

Proof. After using the binomial theorem to expand (17), cancelling $n!$, and cross-multiplying, we see that (17) is equivalent to

$$\frac{k!(n-k)!}{\lambda!(n-\lambda)!} \geq \frac{n-\lambda+1}{n-2\lambda+1}. \tag{18}$$

Using (16) to expand the right-hand side of (18), we see that

$$\frac{n-\lambda+1}{n-2\lambda+1} \leq \frac{(n/2)+(\sqrt{n+2}/2)+1}{\sqrt{n+2}-2+1} = \frac{1}{2}\left(\frac{n+\sqrt{n+2}+2}{\sqrt{n+2}-1}\right) \leq \sqrt{n} \tag{19}$$

for sufficiently large n. It is therefore enough to show that

$$\frac{k!(n-k)!}{\lambda!(n-\lambda)!} \geq \sqrt{n} \tag{20}$$

for sufficiently large n.

Next, since

$$\frac{k!(n-k)!}{\lambda!(n-\lambda)!} = \frac{(n-k)(n-k-1)\cdots(n-\lambda+1)}{\lambda(\lambda-1)\cdots(k+1)}, \tag{21}$$

applying the lower and upper estimates of Lemma 3.3 to the numerator and denominator of the right-hand side of (21), respectively, we see that

$$\begin{aligned} \frac{k!(n-k)!}{\lambda!(n-\lambda)!} &\geq \left[\frac{(n-k)(n-\lambda+1)}{((\lambda+k+1)/2)^2}\right]^{(\lambda-k)/2} \\ &= \left[\frac{4(n-k)(n-\lambda+1)}{(\lambda+k+1)^2}\right]^{(\lambda-k)/2}. \end{aligned} \tag{22}$$

Furthermore, by (16), we see that

$$\begin{aligned} &\frac{4(n-k)(n-\lambda+1)}{(\lambda+k+1)^2} \\ &\geq \frac{(n+A(n)\sqrt{n+2})(n+\sqrt{n+2})}{(n-\frac{1}{2}(A(n)+1)\sqrt{n+2}+2)^2} \\ &= \frac{n^2+n(A(n)+1)\sqrt{n+2}+(n+2)A(n)}{n^2-n(A(n)+1)\sqrt{n+2}+4n+(\frac{1}{2}(A(n)+1)\sqrt{n+2}-2)^2} \\ &= 1+\frac{2n(A(n)+1)\sqrt{n+2}+O(nA(n)^2)}{n^2+O(n^{3/2}A(n))} \\ &= 1+\xi(n), \end{aligned} \tag{23}$$

where $\xi(n) \sim 2A(n)n^{-1/2}$.

Then, since

$$\lambda - k \geq \frac{1}{2}(A(n)-1)\sqrt{n+2}, \tag{24}$$

we see that

$$\frac{k!(n-k)!}{\lambda!(n-\lambda)!} \geq \left[\frac{4(n-k)(n-\lambda+1)}{(\lambda+k+1)^2}\right]^{(\lambda-k)/2} \geq (1+\xi(n))^{\frac{1}{4}(A(n)-1)\sqrt{n+2}}. \quad (25)$$

Therefore, for sufficiently large n, we may apply Lemma 4.1 to obtain (20). The theorem follows. □

Combining Theorem 4.2 with Lemma 3.4, we see immediately that:

Corollary 4.3. *If $\lambda(n)$ and $k(n)$ are defined by* (16)*, then for sufficiently large n,*

$$\binom{n}{\lambda(n)-j} \geq \binom{n}{\lambda(n)-j-1} + \binom{n}{\ell} \quad (26)$$

for $0 \leq j \leq k(n)$ and $\ell \leq k(n)-j$. □

Remark 4.4. As we shall see, to get closer to answering Füredi's question, we would like to make $A(n)$ smaller, or ideally, constant. However, for the right-hand side of (15) to be at least $(1/2)\log n$, we need $A(n)$ to be larger than $\sqrt{\log n}$. Therefore, at least with the above estimation methods, it seems that $c\sqrt{\log n}$, with $c > 1$, is the best possible value of $A(n)$.

5. Proof of the main theorem

Fix $c > 1$, let $A(n)$, $\lambda = \lambda(n)$, and $k = k(n)$ be defined by (11) and (16), and let

$$e(n) = n - \lfloor n/2 \rfloor - \lambda(n) + 1, \qquad f(n) = n - 2k(n) - 1. \quad (27)$$

Note that $e(n) \sim \frac{1}{2}\sqrt{n}$ and $f(n) \sim c\sqrt{n \log n}$. Therefore, to prove the Main Theorem, it is enough to show that:

Theorem 5.1. *There exists an integer N (depending only on c) such that for all $n > N$, there is a decomposition of the Boolean lattice $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, all of which have size between $e(n)$ and $f(n)$.*

Before proving Theorem 5.1, we summarize the main theorem of [7]. Recall that the symmetric chain decomposition of $\mathbf{2}^{[n]}$ given by de Bruijn, Tengbergen, and Kruyswijk [2] is called the *canonical symmetric chain decomposition* (CSCD) of $\mathbf{2}^{[n]}$ (see [7, Sect. 2] for more on the CSCD). Then by [7, Sect. 6], we have that:

Theorem 5.2. *For $n \geq 0$, we may partition $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains of size at least $e(n)$, by starting with the CSCD and repartitioning the portion of $\mathbf{2}^{[n]}$ contained in chains of size at most $n - 2\lambda(n) + 1$ (i.e., chains whose tails are at level r such that $\lambda(n) \leq r \leq \lfloor n/2 \rfloor$). Furthermore, this repartition does not alter the portion of $\mathbf{2}^{[n]}$ contained in longer chains of the CSCD.* □

Proof of Theorem 5.1. Since the partition in Theorem 5.2 is obtained by rearranging only the "short" chains of the CSCD, to prove Theorem 5.1, it is enough to rearrange only the "long" chains of the CSCD into a partition with the same number of chains, but with the size of all chains less than the desired maximum.

Our strategy for doing so is shown in Figure 1, and can be described as follows. Starting with the CSCD, we divide levels 0 through k of $\mathbf{2}^{[n]}$ into "icicles", as shown on the left-hand side of Figure 1, and then use Corollary 2.3 to attach the icicles to the "overhangs" between levels λ and $\lambda-1$, levels $\lambda-1$ and $\lambda-2$, and so on, as shown on the right-hand side of Figure 1. (In this context, the point of Theorem 4.2 is to find a k low enough to allow level k to fit into the overhang between levels λ and $\lambda-1$.) By symmetry, we may also perform the analogous operation on the upper half of $\mathbf{2}^{[n]}$, obtaining a chain partition with the desired maximum chain size.

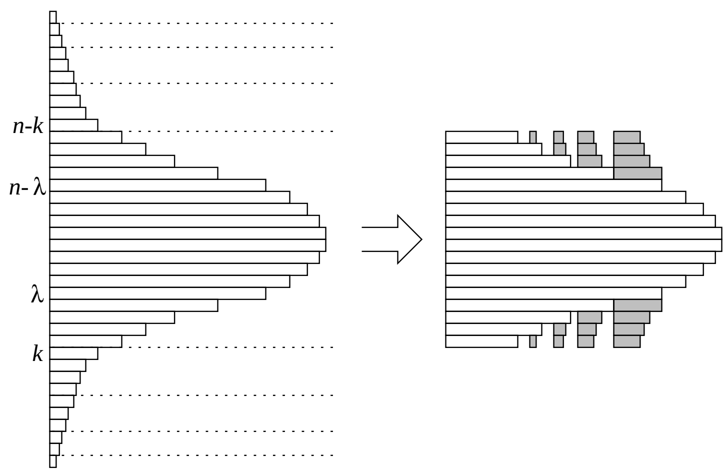

Figure 1: Hanging icicles

More precisely, let n be large enough that the conclusions of Theorem 4.2 and Corollary 4.3 hold. We start with the CSCD, and first rearrange the "short" chains as described in Theorem 5.2. (Note that the rearranging of the "short" chains is not shown in Figure 1.) Next, we perform the following "icicle-hanging" procedure.

1. First, by Theorem 4.2, level λ is larger than the union of levels $\lambda-1$ and k, so by Corollary 2.3, there exists a matching from the union of levels $\lambda-1$ and k to a subset of level λ. We use this matching to hang both the CSCD chains between levels $\lambda-1$ and $k+1$ and the CSCD chains between levels k and $k-(\lambda-k-1)+1$ off of level λ. In other words, we attach $\lambda-k-1$ levels of icicles to the overhang between levels λ and $\lambda-1$.

   Note that all of these new chains will now stop at or above level $k+1$, and that each of the chains of the CSCD between levels $\lambda-1$ and $k+1$ is left intact. Note also, however, that each chain of the CSCD between levels $\lambda - 1$ and $k + 1$ is no longer necessarily attached to the same node in level λ as before. (See Remark 5.3.)

2. Next, note that $\ell = k - (\lambda - k - 1)$ is the first level of icicles that has not yet been attached. Since $\ell \leq k - 1$, Corollary 4.3 implies that level $\lambda - 1$ is larger than the union of levels $\lambda - 2$ and ℓ, so by Corollary 2.3, there exists a matching from the union of levels $\lambda - 2$ and ℓ to a subset of level $\lambda - 1$. We use this matching to attach $\lambda - k - 2$ levels of icicles to the overhang between levels $\lambda - 1$ and $\lambda - 2$. Note that all of these new chains will still stop at or above level $k + 1$.

3. We continue this process, attaching $\lambda - k - 3$ levels of icicles to the overhang between levels $\lambda - 2$ and $\lambda - 3$, $\lambda - k - 4$ levels of icicles to the overhang between levels $\lambda - 3$ and $\lambda - 4$, and so on, either until there are no more icicle levels between 0 and k left to attach, or until we attach 1 level of icicles to the overhang between levels $k + 2$ and $k + 1$. Note that since the top icicle level always descends by at least one level each time the overhang level descends by one level, Corollary 4.3 implies that the size of the top icicle level will always be smaller than the size of the corresponding overhang, which means that we can always apply Corollary 2.3 to obtain the desired matchings.

Now, in the above process, we attach at most $(\lambda - k - 1) + (\lambda - k - 2) + \cdots + 1$ levels of icicles, coming from levels k through 0, to overhangs. Therefore, all of $\mathbf{2}^{[n]}$ will be included in this new chain partition exactly if $\sum_{i=1}^{\lambda-k-1} i \geq k + 1$. However,

$$\begin{aligned} \sum_{i=1}^{\lambda-k-1} i &= \frac{(\lambda - k)(\lambda - k - 1)}{2} \\ &\geq \frac{1}{2}\left(\frac{1}{2}(A(n) - 1)\sqrt{n+2} - 1\right)^2 \qquad (28) \\ &\sim \left(\frac{c^2}{8}\right) n \log n, \end{aligned}$$

by (24). Therefore, since $k(n) \leq n/2$, for sufficiently large n, all of $\mathbf{2}^{[n]}$ will be included.

Finally, by well-known symmetry properties of $\mathbf{2}^{[n]}$, we perform the same procedure on the upper parts of the long chains of the CSCD (levels $n - \lambda$ and up) that we did on the lower parts of the CSCD, obtaining a new chain partition $\mathbf{C}$. We now observe that:

1. The chains of $\mathbf{C}$ correspond bijectively with the chains of the CSCD.

2. None of the chains of $\mathbf{C}$ modified in Theorem 5.2 are shortened in the icicle-hanging process. Furthermore, each of the icicle-hanging chains is obtained by lengthening either a chain modified in Theorem 5.2 or a chain that has

an element at each level between λ and $n-\lambda$. Therefore, each of the chains of $\mathbf{C}$ has size at least $e(n) < n - 2\lambda$.

3. Since each chain of $\mathbf{C}$ can be thought of as fitting between levels $k+1$ and $n-(k+1)$, each chain of $\mathbf{C}$ has size at most $n - 2(k+1) + 1 = f(n)$.

Theorem 5.1 follows. □

Remark 5.3. We remark that Figure 1 is somewhat misleading, as it implies that the icicle hanging from level λ is attached to precisely those nodes of level λ that are tails of chains of the CSCD. However, since the tails of the CSCD form an ideal (or downset) of $\mathbf{2}^{[n]}$ (case $j = 0$ of [7, Cor. 4.3]), this *cannot* occur.

What actually happens is that (for example) when we take the icicle starting at level k and attach it to level λ, this icicle gets mixed in with the portion of the chains of the CSCD that runs between levels $\lambda - 1$ and $k+1$. Nevertheless, since nothing new is hung from level $\lambda - 1$, we may continue to attach icicles to level $\lambda - 1$, level $\lambda - 2$, and so on, as described above.

We imagine that results of this "mixed icicle-hanging" might look something like the left-hand side of Figure 2 (as divided by the dotted line). Also, to illustrate the final result of Theorem 5.1, we give a rough picture of the lengthening process of Theorem 5.2 on the right-hand side of Figure 2. (See [7] for a more accurate picture, and details.) Note that although the lengthening process of Theorem 5.2 and the icicle-hanging process in the proof of Theorem 5.1 actually both involve nodes in level λ, for the sake of clarity, we have not shown the interaction between the two processes. For a description how this interaction works, see observation (2) at the end of the proof of Theorem 5.1.

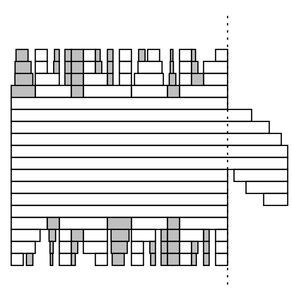

Figure 2: Mixed icicle-hanging and chain lengthening

Remark 5.4. For the reader interested in how the constant N in Theorem 5.1 depends on the choice of $c > 1$, we note that to obtain the result of Theorem 5.1, n need only be large enough for (17) and (28) to hold. Curiously, although the proof of Theorem 4.2 is much more involved than the proof of (28), a computer check shows that even for $c = 1$ (the limiting case for our purposes), (17) seems to hold for all n, whereas for $c = 1.5$, (28) holds for $n > 16$, and for $c = 1$, (28) only holds for $n > 6990$.

References

[1] I. Anderson. *Combinatorics of Finite Sets*. Oxford Univ. Press, 1987.

[2] N. G. de Bruijn, C. Tengbergen, and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wiskd.*, 23:191–193, 1951.

[3] K. Engel. *Sperner Theory*. Cambridge Univ. Press, 1997.

[4] Z. Füredi. Problem session. In *Kombinatorik geordneter Mengen*, Oberwolfach, B. R. D., 1985.

[5] J. R. Griggs. Problems on chain partitions. *Disc. Math.*, 72:157–162, 1988.

[6] J. R. Griggs. Matchings, cutsets, and chain partitions in graded posets. *Disc. Math.*, 144:33–46, 1995.

[7] T. Hsu, M. J. Logan, S. Shahriari, and C. Towse. Partitioning the Boolean lattice into chains of large minimum size. *J. Comb. Thy. (A)*, 97(1):62–84, 2002.