Partitioning the Boolean lattice into chains of large minimum size

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Key Words: Boolean lattice, chain decompositions, Füredi's problem, Griggs' conjecture, normalized matching property

Let $\mathbf{2}^{[n]}$ denote the *Boolean lattice* of order n, that is, the poset of subsets of $\{1, \ldots, n\}$ ordered by inclusion. Recall that $\mathbf{2}^{[n]}$ may be partitioned into what we call the *canonical symmetric chain decomposition* (due to de Bruijn, Tengbergen, and Kruyswijk), or CSCD.

Motivated by a question of Füredi, we show that there exists a function $d(n) \sim \frac{1}{2}\sqrt{n}$ such that for any $n \geq 0$, $\mathbf{2}^{[n]}$ may be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains of size at least $d(n)$. (For comparison, a positive answer to Füredi's question would imply that the same result holds for some $d(n) \sim \sqrt{\pi/2}\sqrt{n}$.) More precisely, we first show that for $0 \leq j \leq n$, the union of the lowest $j+1$ elements from each of the chains in the CSCD of $\mathbf{2}^{[n]}$ forms a poset $\mathbf{T}_j(n)$ with the normalized matching property and log-concave rank numbers. We then use our results on $\mathbf{T}_j(n)$ to show that the nodes in the CSCD chains of size less than $2d(n)$ may be repartitioned into chains of large minimum size, as desired.

1. INTRODUCTION

Let $[n] = \{1, \ldots, n\}$ and let $\mathbf{2}^{[n]}$ denote the Boolean lattice of order n, that is, the poset of subsets of $[n]$ ordered by inclusion. A collection of subsets $A_0 \subset \cdots \subset A_k$ of $[n]$ is called a chain of size $k+1$ (or length k) in $\mathbf{2}^{[n]}$. In this paper we construct a partition of $\mathbf{2}^{[n]}$ into a collection of chains such that the size of the shortest chain is approximately $\frac{1}{2}\sqrt{n}$.

More precisely, let

$$d(n) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} - \frac{\sqrt{n+2}}{2} \right\rceil, \tag{1}$$

$$\epsilon(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

(The significance of $d(n)$ and $\epsilon(n)$ will be made clearer later; for the moment, it is enough to note that $d(n) \sim \frac{1}{2}\sqrt{n}$ for large n.) Our main result is:

Main Theorem. *For any $n \geq 0$, the Boolean lattice $\mathbf{2}^{[n]}$ may be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains of size at least $d(n) + \epsilon(n)$.*

Note that in any chain partition of $\mathbf{2}^{[n]}$, no two subsets of size $\lfloor n/2 \rfloor$ can be in the same chain, so $\binom{n}{\lfloor n/2 \rfloor}$ is the smallest possible number of chains in any chain partition of $\mathbf{2}^{[n]}$. Note also that the Main Theorem immediately implies:

Corollary. *For any $n \geq 0$, the Boolean lattice $\mathbf{2}^{[n]}$ may be partitioned into chains of size between $d(n) + \epsilon(n)$ and $2(d(n) + \epsilon(n))$.* ▮

Our Main Theorem is motivated by several well-known questions on chain partitions of the Boolean lattice, the most directly relevant of which is due to Füredi [6]. Such questions began with Sands [16], who asked if, for a given k and for large enough n, $\mathbf{2}^{[n]}$ can be partitioned into chains of size 2^k. More generally, Griggs [9] later conjectured that for a given $c \geq 1$ and for n sufficiently large, $\mathbf{2}^{[n]}$ can be partitioned into chains of size c and a single chain of size at most $c-1$, a conjecture later proven by Lonc [15] (see below).

Moving in another direction, Füredi [6] asked if $\mathbf{2}^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains such that the size of every chain is one of two consecutive integers. In other words, if we define the integers $a(n)$ and $b(n)$

by

$$2^n = a(n)\binom{n}{\lfloor n/2\rfloor} + b(n), \qquad 0 \le b(n) < \binom{n}{\lfloor n/2\rfloor}, \tag{3}$$

then Füredi's question is: Can $\mathbf{2}^{[n]}$ can be partitioned into $b(n)$ chains of size $a(n)+1$ and $\binom{n}{\lfloor n/2\rfloor} - b(n)$ chains of size $a(n)$?

The most general question of this type is due to Griggs [9], who asked: Given a partition $\boldsymbol{\mu} = (\mu_1 \ge \cdots \ge \mu_\ell)$ of 2^n into positive parts, is there a partition of $\mathbf{2}^{[n]}$ into chains of sizes $\mu_1, \ldots, \mu_\ell$? In fact, Griggs also conjectured an answer to this question, the statement of which requires the following terminology. Recall that a chain $A_0 \subset \cdots \subset A_k$ in $\mathbf{2}^{[n]}$ is *skipless*, or *saturated*, if, for $1 \le i \le k$, we have $|A_{i-1}| + 1 = |A_i|$; that a skipless chain $A_0 \subset \cdots \subset A_k$ in $\mathbf{2}^{[n]}$ is said to be *symmetric* if $|A_0| + |A_k| = n$; that a partition of $\mathbf{2}^{[n]}$ into symmetric chains is called a *symmetric chain decomposition*, or SCD, of $\mathbf{2}^{[n]}$; and that for every $n \ge 0$, $\mathbf{2}^{[n]}$ has an SCD (see Section 2). Recall also that if $\sigma_k = n - 2j + 1$ for all k and j such that $\binom{n}{j-1} < k \le \binom{n}{j}$ and $0 \le j \le \lfloor n/2\rfloor$, then the partition

$$\boldsymbol{\sigma} = (\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_{\binom{n}{\lfloor n/2\rfloor}}) \tag{4}$$

of 2^n, which we call the *SCD partition*, has the property that the chains in any symmetric chain decomposition of $\mathbf{2}^{[n]}$ have sizes σ_1, σ_2, and so on. Finally, recall that the *dominance* (or *majorization*) order on partitions of an integer m is defined by the rule that for partitions $\boldsymbol{\mu} = (\mu_i)$ and $\boldsymbol{\nu} = (\nu_i)$ of m, we have that $\boldsymbol{\mu} \le \boldsymbol{\nu}$ if and only if for all j,

$$\sum_{i=1}^{j} \mu_i \le \sum_{i=1}^{j} \nu_i. \tag{5}$$

Griggs' conjecture [9] is that for a partition $\boldsymbol{\mu}$ of the integer 2^n, there exists a partition of $\mathbf{2}^{[n]}$ into chains with sizes given by $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu} \le \boldsymbol{\sigma}$ in the dominance order.

It is worth noting that answering Füredi's question is almost certainly an important first step to proving Griggs' conjecture, in the following sense. Consider those partitions of 2^n that are dominated by the SCD partition and have exactly $\binom{n}{\lfloor n/2\rfloor}$ positive parts. This subposet of the lattice of partitions (with respect to majorization) has a unique maximal element, the SCD partition, and a unique minimal element, the Füredi partition. It is not hard to see that in this subposet, the partitions that are "close" to the

SCD partition can be obtained by modifying the symmetric chain decomposition of $\mathbf{2}^{[n]}$. Conversely, it seems that the partitions that are "close" to the Füredi partition (and therefore, "far" from the SCD partition) are the part of Griggs' conjecture that is least well-understood.

Now, while Füredi's question and Griggs' conjecture are certainly still open, progress has been made by several authors. For example, Griggs, Grinstead, and Yeh [12] proved that $\mathbf{2}^{[n]}$ can be partitioned into chains of size 4 if and only if $n \geq 9$. Griggs [10] later modified the standard inductive construction of a symmetric chain decomposition of $\mathbf{2}^{[n]}$ (see Section 2) to create a chain decomposition of $\mathbf{2}^{[n]}$ with a large number of chains of size at most c. In particular, applying this construction, and thinking of the desired chain size c as a function of n, if $c(n) = o(\sqrt{n})$, the proportion of subsets in $[n]$ that belong to chains of size c approaches 1 as $n \to \infty$, and if $c(n) \sim a\sqrt{n}$ for some constant a, the same proportion approaches a constant that depends on a and is strictly less than 1.

Perhaps the most significant progress to date was made by Lonc [15], who showed that for a given constant $c \geq 1$ and n sufficiently large, $\mathbf{2}^{[n]}$ can be partitioned into chains of size c and a single chain of size at most $c - 1$, thus proving the first of Griggs' conjectures mentioned above. Note, however, that in Lonc's proof, n is required to be quite large, as a function of c; in fact, for a given c, it is not hard to see that $2^{\exp(c^2)}$ is a (coarse) lower bound for the required size of n. In other words, for a given n, only a relatively small desired chain size c can be achieved.

As a measure of the progress that we have made in this paper towards answering Füredi's question, note that in the (hypothetical) Füredi partition, the number of chains is $\binom{n}{\lfloor n/2 \rfloor}$, and the sizes of the chains are $a(n)$ and $a(n) + 1$, where $a(n)$ is given by (3). Since Stirling's approximation then implies that $a(n) \sim \sqrt{\pi/2}\sqrt{n}$, we see that our Main Theorem gives a chain partition with the correct number of chains, and with minimal chain size roughly $\sqrt{1/(2\pi)} \approx 39.89\%$ of $a(n)$. In other words, in some sense, we have answered just under 40% of Füredi's question. On the other hand, to achieve the Füredi partition, we would have to increase the size of the shortest chain to $a(n)$, while at the same time decreasing the size of the longest chain to $a(n) + 1$. Satisfying these more stringent conditions, especially the latter, will require ideas beyond the current paper. (See Question 7.5.)

We now briefly summarize the rest of this paper. Our basic strategy is to start with what we call the *canonical symmetric chain decomposition*, or CSCD, of $\mathbf{2}^{[n]}$ (Section 2), and then modify it to obtain the desired chain partition. This modification may be described heuristically as follows. Recall that for large n, if we draw the chains in the CSCD in order of decreasing size, from left to right, on their appropriate levels, then we

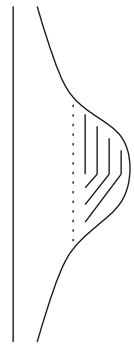

FIG. 1. Sketch of our basic strategy

obtain a diagram that resembles a Gaussian distribution, as shown in Figure 1. Our modification of the CSCD only affects the portion of the Boolean lattice that lies to the right of the "inflection point" of this distribution; in other words, as we shall see, we only modify the chains in the CSCD of size less than (roughly) $\sqrt{n}$. In fact, even in these modified "short" chains, we only rearrange the bottom halves of the chains, leaving the parts of the chains above the middle level $\lfloor n/2 \rfloor$ unaffected, as sketched in Figure 1.

More precisely, let $\mathbf{T}_j(n)$ $(j \geq 0)$ be the union of the lowest $j+1$ elements from each of the chains in the CSCD of $\mathbf{2}^{[n]}$ (Section 3). The key point is that $\mathbf{T}_j(n)$ has a structure that greatly resembles $\mathbf{2}^{[n]}$ itself; more specifically, we show that $\mathbf{T}_j(n)$ is a graded simplicial complex (also called a downset, or ideal) with the normalized matching property and log-concave rank numbers (Section 4). As a consequence, after a closer examination of the rank numbers of $\mathbf{T}_j(n)$ (Section 5), we are able to modify the CSCD to achieve the partition described in the Main Theorem (Section 6). We conclude by discussing some open questions (Section 7).

2. BACKGROUND ON THE BOOLEAN LATTICE

In this section, we review some facts about, and introduce some notation and terminology for, the Boolean lattice $\mathbf{2}^{[n]}$.

Notation 2.1. Throughout this paper, $[n]$ denotes the set $\{1, \ldots, n\}$ ($[0] = \emptyset$) and $\mathbf{2}^{[n]}$ denotes the Boolean lattice of subsets of $[n]$. Depending on the context, we refer interchangably to subsets of $[n]$ and nodes of $\mathbf{2}^{[n]}$, and we denote inclusion, or the partial order, by either $\subseteq$ or $\leq$. Similarly, when we wish to consider $\mathbf{2}^{[n]}$ as a graph, we refer to its Hasse diagram as $\mathbf{2}^{[n]}$.

Definition 2.2. Recall that $\mathbf{2}^{[n]}$ is a *ranked* poset, with level k of $\mathbf{2}^{[n]}$ defined to be the set of all $A \in \mathbf{2}^{[n]}$ such that $|A| = k$. If $A_0 < \cdots < A_k$

is a chain C in $\mathbf{2}^{[n]}$, then A_0 is called the *tail* of C, and A_k is called the *head* of C. The *shadow* of $A \in \mathbf{2}^{[n]}$ is the set of all $A^- < A$ such that $|A^-| = |A| - 1$, and the *shade* of $A \in \mathbf{2}^{[n]}$ is the set of all $A^+ > A$ such that $|A^+| = |A| + 1$.

We next define what we call the *canonical symmetric chain decomposition* of $\mathbf{2}^{[n]}$, as constructed by de Bruijn, Tengbergen, and Kruyswijk [4].

Definition 2.3. The *canonical symmetric chain decomposition*, or *CSCD*, of $\mathbf{2}^{[n]}$ is given by the following recursive definition:

1. The CSCD of $\mathbf{2}^{[0]}$ contains the single chain $\emptyset$.
2. For $n \geq 1$, the CSCD of $\mathbf{2}^{[n]}$ contains precisely the following chains.

(i) For every chain $A_0 < \cdots < A_k$ in the CSCD of $\mathbf{2}^{[n-1]}$ with $k > 0$, the CSCD of $\mathbf{2}^{[n]}$ contains the chains

$$A_0 < \cdots < A_k < A_k \cup \{n\} \text{ and } A_0 \cup \{n\} < \cdots < A_{k-1} \cup \{n\}\,.$$

(ii) For every chain A_0 of size 1 in the CSCD of $\mathbf{2}^{[n-1]}$, the CSCD of $\mathbf{2}^{[n]}$ contains the chain $A_0 < A_0 \cup \{n\}$.

Both cases of this description are illustrated in Figure 2.

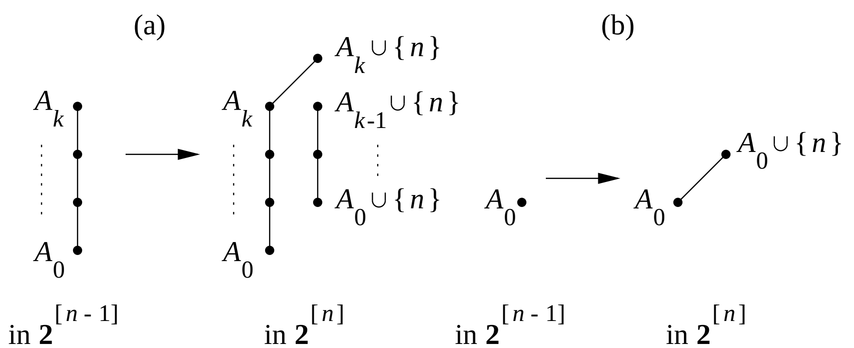

FIG. 2. Recursive desciption of CSCD chains

Example 2.4. Figure 3 (resp. Figure 4) describes the CSCD of $\mathbf{2}^{[4]}$ (resp. $\mathbf{2}^{[5]}$) by displaying the elements of $\mathbf{2}^{[4]}$ (resp. $\mathbf{2}^{[5]}$) in columns that correspond to the chains of the CSCD. As an exercise, the reader who is less familar with the CSCD may wish to derive Figure 4 from Figure 3.

For more on symmetric chain decompositions of $\mathbf{2}^{[n]}$ and other posets, see Anderson [2, Chap. 3].

1234					
123	234	134	124		
12	23	13	14	24	34
1	2	3	4		
$\emptyset$					

FIG. 3. The CSCD of $\mathbf{2}^{[4]}$

12345									
1234	2345	1345	1245	1235					
123	234	134	124	125	235	135	145	245	345
12	23	13	14	15	25	35	45	24	34
1	2	3	4	5					
$\emptyset$									

FIG. 4. The CSCD of $\mathbf{2}^{[5]}$

3. DECOMPOSITION OF THE CSCD BY TAIL-HEIGHT

In this section, we examine the chains of the CSCD of $\mathbf{2}^{[n]}$, we find that the structure of $\mathbf{2}^{[n]}$ decomposes nicely relative to the height of a node above the tail of its chain in the CSCD, and we obtain some preliminary results about this decomposition.

Definition 3.1. For $A \in \mathbf{2}^{[n]}$, if the chain containing A in the CSCD of $\mathbf{2}^{[n]}$ is $A_0 < \cdots < A_\ell$, and $A = A_h$, we say that A has *tail-height* h. Note that by definition, tail-height is always nonnegative, so we will always assume that any variable referring to tail-height is nonnegative.

Definition 3.2. We define $t_{h,k}(n)$ to be the set of all nodes of $\mathbf{2}^{[n]}$ at level k and tail-height h. We also define

$$T_{j,k}(n) = \bigcup_{h=0}^{j} t_{h,k}(n), \tag{6}$$

$$\mathbf{T}_j(n) = \bigcup_{k=0}^{n} T_{j,k}(n). \tag{7}$$

In other words, $T_{j,k}(n)$ is the set of all nodes at level k and tail-height at most j in $\mathbf{2}^{[n]}$, and $\mathbf{T}_j(n)$ is the set of all nodes with tail-height at most j in $\mathbf{2}^{[n]}$.

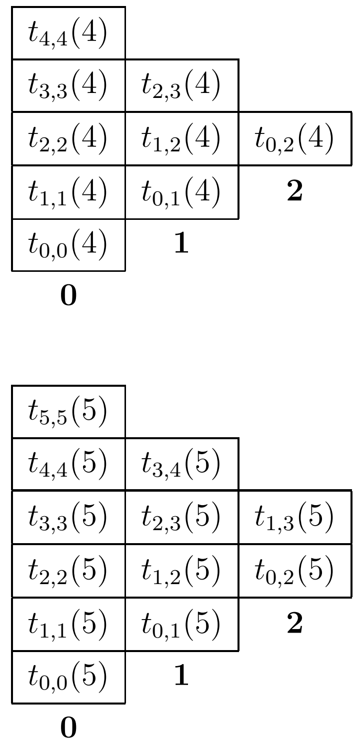

FIG. 5. Tail-height/level coordinates (h, k) for $\mathbf{2}^{[4]}$ and $\mathbf{2}^{[5]}$

We may think of tail-height and level as forming a "coordinate system" for $\mathbf{2}^{[n]}$ based on the CSCD. This system is shown schematically for $n = 4, 5$ in Figure 5. Each column of Figure 5 represents all chains of the CSCD of size $n - 2(k - h) + 1$, where $k - h$ is constant along the column and is indicated by the boldface number at the bottom of the column. For comparison, returning to Figures 3 and 4 of Example 2.4, we see that the boxes in Figure 3 (resp. Figure 4) show how the actual elements of $\mathbf{2}^{[4]}$ (resp. $\mathbf{2}^{[5]}$) decompose into $t_{h,k}(n)$'s, in a manner corresponding to the boxes in the diagrams in Figure 5.

More formally, we begin with the following basic observations about tail-height, level, and the CSCD.

Lemma 3.3. *For $n \geq 0$, let A be a node in $\mathbf{2}^{[n]}$ at level k and tail-height h, and let C be the CSCD chain containing A. Then $0 \leq k - h \leq \lfloor n/2 \rfloor$,* *the head of* C *is at level* $n-(k-h)$, C *has size* $n-2(k-h)+1$, *and* A *is the head of* C *if and only if* $n-2k+h=0$.

Proof. By the definition of tail-height, the tail of C is at level $k-h$, and by the symmetry of C, the head of C is at level $n-(k-h)$. The rest of the lemma follows easily. (The reader may wish to compare Figure 5, especially to verify that the heads satisfy $n-2k+h=0$.) ∎

We finish this section by describing the sizes of the $t_{h,k}(n)$ and $T_{j,k}(n)$. Towards this end, it will be convenient to have the following function.

Definition 3.4. We define $\delta(n,a,b)=\binom{n}{a}-\binom{n}{b}$.

Lemma 3.5. *Fix* $n\geq 0$.

1. *For* $h\geq 0$, $t_{h,k}(n)$ *is nonempty if and only if* $h\leq k\leq \min(n,(n+h)/2)$.
2. *For* $j\geq 0$, $T_{j,k}(n)$ *is nonempty if and only if* $0\leq k\leq \min(n,(n+j)/2)$.
3. *For* $h,j\geq 0$, *when* $t_{h,k}(n)$ *is nonempty,* $|t_{h,k}(n)|=\delta(n,k-h,k-h-1)$, *and when* $T_{j,k}(n)$ *is nonempty,* $|T_{j,k}(n)|=\delta(n,k,k-j-1)$.

Proof. Beginning with part (1.), on the one hand, suppose A is a node at level k and tail-height h contained in a CSCD chain C. In that case, since $k-h$ is the level of the tail of C, $k-h\geq 0$, and since k cannot be greater than either n or the level of the head of C, $k\leq\min(n,(n+h)/2)$ (using Lemma 3.3). On the other hand, suppose we have $h\geq 0$ and $h\leq k\leq\min(n,(n+h)/2)$. Then, since $0\leq k-h\leq\lfloor n/2\rfloor$, by well-known properties of the CSCD, there exist tails of chains of the CSCD at level $k-h$, and since $k-h\leq k\leq n-(k-h)$, by Lemma 3.3, there exist nodes at level k in the CSCD chains above those tails. Part (1.) follows.

Turning to part (2.), we see that it is enough to show, for $0\leq k\leq n$, that $T_{j,k}(n)$ is nonempty if and only if $k\leq (n+j)/2$. In that case, from part (1.), we know, for fixed k, that $t_{h,k}(n)$ is nonempty if and only if $2k-n\leq h\leq k$. It follows from the definition of $T_{j,k}(n)$ (Definition 3.2) that

$$T_{j,k}(n)=\bigcup_{h=0}^{j} t_{h,k}(n)=\bigcup_{h=2k-n}^{j} t_{h,k}(n), \tag{8}$$

which is nonempty if and only if $j\geq 2k-n$. Part (2.) follows.

Finally, when $t_{h,k}(n)$ is nonempty, the chains of the CSCD give a perfect matching between $t_{h,k}(n)$ and $t_{0,k-h}(n)$, which means that $|t_{h,k}(n)|=|t_{0,k-h}(n)|$. Then, since $|t_{0,k-h}(n)|$ is just the number of chains in the CSCD whose tails are at level $k-h$, we have that

$$|t_{0,k-h}(n)|=\binom{n}{k-h}-\binom{n}{k-h-1}=\delta(n,k-h,k-h-1). \tag{9}$$

When $T_{j,k}(n)$ is nonempty, summing (9) from $h = 2k - n$ to $h = j$, noting that $k - (2k - n) = n - k$, and collapsing the sum, we see that

$$\begin{aligned} |T_{j,k}(n)| &= \binom{n}{n-k} - \binom{n}{n-k-1} + \cdots + \binom{n}{k-j} - \binom{n}{k-j-1} \\ &= \binom{n}{n-k} - \binom{n}{k-j-1} = \binom{n}{k} - \binom{n}{k-j-1} \\ &= \delta(n, k, k-j-1). \end{aligned} \tag{10}$$

The lemma follows. ▮

4. THE STRUCTURE OF $\mathbf{T}_j(n)$

We now come to the key results of this paper. Briefly, we show that, roughly speaking, $\mathbf{T}_j(n)$ has a structure much like that of the Boolean lattice itself (Theorem 4.2 and its corollaries). We begin with the following definition.

Definition 4.1. If $\{A_i\}$ is a collection of subsets of $[n-1]$, we define $\{A_i\} * \{n\}$ to be the collection $\{A_i \cup \{n\}\}$ of subsets of $[n]$.

We come to the following theorem, which shows that the levels of $\mathbf{T}_j(n)$ have nearly the same recursive structure as the levels of $\mathbf{2}^{[n]}$ itself.

Theorem 4.2. *For $j \geq 0$, $n \geq 1$, we have that*

$$T_{j,k}(n) = \begin{cases} T_{j,k}(n-1) \cup (T_{j,k-1}(n-1) * \{n\}) & \text{if } 0 \leq k \leq (n+j)/2, \\ \emptyset & \text{otherwise.} \end{cases} \tag{11}$$

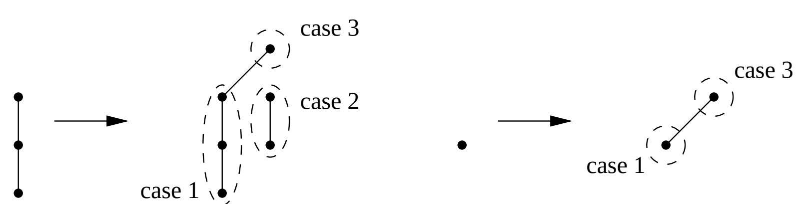

FIG. 6. Cases in proof of Theorem 4.2

Proof. If $k < 0$ or $k > n$, then both sides of (11) are empty, and if $k > (n+j)/2$, then Lemma 3.5, part (2.), implies that $T_{j,k}(n) = \emptyset$. It therefore remains to consider the case $0 \leq k \leq \min(n, (n+j)/2)$. (In particular, we may assume that $n - 2k + j \geq 0$.)

We first show that $T_{j,k}(n-1) \cup (T_{j,k-1}(n-1) * \{n\}) \subseteq T_{j,k}(n)$. By the recursive definition of the CSCD (Definition 2.3), every chain in the CSCD of $\mathbf{2}^{[n-1]}$ is the bottom of a chain in the CSCD of $\mathbf{2}^{[n]}$. Therefore, $T_{j,k}(n-1) \subseteq T_{j,k}(n)$, and it remains only to show that $T_{j,k-1}(n-1) * \{n\} \subseteq T_{j,k}(n)$. So let A be an element of $T_{j,k-1}(n-1) * \{n\}$, let $A_1 = A - \{n\}$, let C_1 be the chain in the CSCD of $\mathbf{2}^{[n-1]}$ that contains A_1, and let $h \leq j$ be the tail-height of A_1 in $\mathbf{2}^{[n-1]}$. On the one hand, if $h < j$, the recursive definition of the CSCD implies that the tail-height of A in $\mathbf{2}^{[n]}$ is at most j. On the other hand, if $h = j$, it follows from Lemma 3.3 that A_1 is not the head of C_1, since

$$(n-1) - 2(k-1) + h = (n-1) - 2(k-1) + j = n - 2k + j + 1 > 0. \tag{12}$$

Therefore, by the recursive definition of the CSCD, the tail-height of A is equal to j (the tail-height of A_1). In either case, $A \in T_{j,k}(n)$, and the first claimed inclusion follows.

Conversely, we must also show that $T_{j,k}(n) \subseteq T_{j,k}(n-1) \cup (T_{j,k-1}(n-1) * \{n\})$. Let A be an element of $T_{j,k}(n)$, and let C be the chain in the CSCD of $\mathbf{2}^{[n]}$ containing A. Now, by the recursive definition of the CSCD, we have three cases:

1. $n \notin A$.
2. Every set in C contains n.
3. A is the head of C and no other sets in C contain n.

These cases are illustrated in Figure 6.

Now, in case 1., we know that C consists of the same nodes as the chain in the CSCD of $\mathbf{2}^{[n-1]}$ that contains A, with a new head adjoined, which means that $A \in T_{j,k}(n-1)$. In case 2., let $A_1 = A - \{n\}$, let C_1 be the chain of the CSCD of $\mathbf{2}^{[n-1]}$ containing A_1, and let $\widehat{A}_1$ be the head of C_1. Note that $A_1 \neq \widehat{A}_1$, that the sets strictly below $\widehat{A}_1$ in C_1 are precisely the sets in C with n removed, and that A_1 is at level $k-1$ in $\mathbf{2}^{[n-1]}$. It follows that the tail-height of A_1 in $\mathbf{2}^{[n-1]}$ is equal to the tail-height of A in $\mathbf{2}^{[n]}$, that $A_1 \in T_{j,k-1}(n-1)$, and that $A \in T_{j,k-1}(n-1) * \{n\}$. Finally, in case 3., let $A_1 = A - \{n\}$ be the head of the chain in the CSCD of $\mathbf{2}^{[n-1]}$ from which C is derived, and let $h \leq j$ be the tail-height of A in $\mathbf{2}^{[n]}$. Then, since A_1 is at level $k-1$ and the tail-height of A_1 in $\mathbf{2}^{[n-1]}$ is $h - 1 \leq j$, it follows that $A = A_1 \cup \{n\} \in T_{j,k-1}(n-1) * \{n\}$. ∎

In the rest of this section, we use Theorem 4.2 to obtain other ways in which $\mathbf{T}_j(n)$ resembles the full Boolean lattice (Corollaries 4.3, 4.4, 4.6, 4.12, and 4.13). We begin with the global structure of $\mathbf{T}_j(n)$ (Corollaries 4.3, 4.4, and 4.6).

Corollary 4.3. *For $j, n \geq 0$, $\mathbf{T}_j(n)$ is a simplicial complex. More precisely, for $j, n \geq 0$ and any k, the shadow of an element of $T_{j,k}(n)$ is contained in $T_{j,k-1}(n)$.*

Proof. For $n = 0$, $k \leq 0$, or $k > (n+j)/2$, the corollary holds vacuously, so proceeding by induction on $n \geq 1$, for $1 \leq k \leq (n+j)/2$, let A be an element of $T_{j,k}(n)$, and let A^- be in the shadow of A. By Theorem 4.2, it suffices to show that

$$A^- \in T_{j,k-1}(n-1) \cup T_{j,k-2}(n-1) * \{n\}\,. \tag{13}$$

Now, again by Theorem 4.2, either $A \in T_{j,k}(n-1)$ or $A \in T_{j,k-1}(n-1)*\{n\}$. On the one hand, if $A \in T_{j,k}(n-1)$, then by induction, $A^- \in T_{j,k-1}(n-1)$. On the other hand, suppose $A \in T_{j,k-1}(n-1) * \{n\}$. In that case, if $A - A^- = \{n\}$, then $A^- \in T_{j,k-1}(n-1)$; and if $A - A^- \neq \{n\}$, then A^- is the union of $\{n\}$ and a member of the shadow of an element of $T_{j,k-1}(n-1)$, which means, by induction, that $A^- \in T_{j,k-2}(n-1)*\{n\}$. The corollary follows. ▮

Recall that a ranked poset is said to be *graded* if every maximal chain has the same size.

Corollary 4.4. *For $j, n \geq 0$, $\mathbf{T}_j(n)$ is a graded poset.*

Proof. Since $\mathbf{T}_j(n)$ is a simplicial complex, and since $\emptyset$ is a node in $\mathbf{T}_j(n)$, it follows easily that a maximal chain in $\mathbf{T}_j(n)$ must be skipless and must extend down to level 0. It is therefore enough to show that a maximal element in $\mathbf{T}_j(n)$ must be an element of maximal level. More precisely, from Lemma 3.5, part (2.), we see that, fixing $j \geq 0$, it suffices to show that for $1 \leq k \leq \min(n, (n+j)/2)$ and $A \in T_{j,k-1}(n)$, there exists some $A^+ \in T_{j,k}(n)$ in the shade of A. This claim is vacuous for $n = 0$, so proceeding by induction on n, let A be an element of $T_{j,k-1}(n)$ for some $n \geq 1$ and $1 \leq k \leq \min(n, (n+j)/2)$. Now, by Theorem 4.2, either $A \in T_{j,k-1}(n-1)$ or $A \in T_{j,k-2}(n-1) * \{n\}$. On the one hand, if $A \in T_{j,k-1}(n-1)$, then we may choose $A^+ = A \cup \{n\} \in T_{j,k-1}(n-1) * \{n\}$. On the other hand, if $A \in T_{j,k-2}(n-1) * \{n\}$, then $k \geq 2$ and

$$1 \leq k-1 \leq \min\left(n-1, \frac{(n-1)+j}{2}\right), \tag{14}$$

which means that by induction, there exists some $A^+ \in T_{j,k-1}(n-1)*\{n\}$ in the shade of A. Either way, since $A^+ \in T_{j,k-1}(n-1)*\{n\} \subseteq T_{j,k}(n)$ (Theorem 4.2), the corollary follows. ▮

Definition 4.5. If P is a ranked poset, we define the *truncation* of P at level m to be the subposet of all elements of P of level at most m. If P and Q are posets, we define the *product poset* $P \times Q$ to be the cartesian product $P \times Q$ with order defined by the rule that $(x, y) \leq (x', y')$ if and only if $x \leq x'$ in P and $y \leq y'$ in Q. Note that if P and Q are ranked posets, then $P \times Q$ is also naturally ranked, with the level of (x, y) defined to be the level of x in P plus the level of y in Q.

Corollary 4.6. *Fix $j \geq 0$, $n \geq 1$, let C be the chain of size 2, and let $P(n)$ denote the product $\mathbf{T}_j(n-1) \times C$.*

1. *If $n \leq j$ or $n + j$ is even, then $\mathbf{T}_j(n)$ is isomorphic to $P(n)$.*
2. *If $n > j$ and $n + j$ is odd, then $\mathbf{T}_j(n)$ is isomorphic to the truncation of $P(n)$ at level $(n + j - 1)/2$. More specifically, $\mathbf{T}_j(n)$ is isomorphic to $P(n)$ with its top level (level $(n + j + 1)/2$) deleted.*

Proof. For $n \leq j$, $\mathbf{T}_j(n) = \mathbf{2}^{[n]}$, and the corollary is just a well-known fact about the Boolean lattice, so without loss of generality, we assume that $n > j$. In that case, let P_k be level k of $P(n)$. Now, from Lemma 3.5, part (2.), we know that the nonempty levels of $\mathbf{T}_j(n-1)$ range between 0 and $\lfloor (n - 1 + j)/2 \rfloor$, inclusive. Therefore, thinking of C as the poset of subsets of $\{n\}$, we may consider $P(n)$ to be a subposet of $\mathbf{2}^{[n]}$ by taking

$$P_k = \begin{cases} T_{j,k}(n-1) & \text{for } k = 0, \\ T_{j,k}(n-1) \cup (T_{j,k-1}(n-1) * \{n\}) & \text{for } 1 \leq k \leq \lfloor (n+j-1)/2 \rfloor, \\ T_{j,k-1}(n-1) * \{n\} & \text{for } k = \lfloor (n+j-1)/2 \rfloor + 1. \end{cases} \tag{15}$$

Comparing (11) from Theorem 4.2, it is clear that $T_{j,k}(n) = P_k$, except possibly for the cases $k = 0$ and $k = \lfloor (n + j - 1)/2 \rfloor + 1$. Now, for $k = 0$, we have that $T_{j,-1}(n - 1)$ is empty and $T_{j,0}(n) = P_0$. Furthermore, if $k = \lfloor (n + j - 1)/2 \rfloor + 1$, then either $n + j$ is even,

$$k = \lfloor (n+j-1)/2 \rfloor + 1 = (n+j)/2 > (n-1+j)/2, \tag{16}$$

$T_{j,k}(n - 1)$ is empty (Lemma 3.5, part (2.)), and $T_{j,k}(n) = P_k$; or $n + j$ is odd,

$$k = \lfloor (n+j-1)/2 \rfloor + 1 = (n+j+1)/2 > (n+j)/2, \tag{17}$$

and $T_{j,k}(n)$ is empty (Lemma 3.5, part (2.)). The corollary follows. ∎

In the rest of this section, we use Corollaries 4.4 and 4.6 to show that $\mathbf{T}_j(n)$ has certain very strong matching properties (Corollaries 4.12 and 4.13), which we define as follows. (For a standard reference on such properties, see Griggs [11].)

Definition 4.7. Let P be a ranked poset. We say that P has the *normalized matching property* if, for any consecutive levels X and Y in P and $Z \subseteq X$, we have

$$\frac{|Z|}{|X|} \leq \frac{|\Gamma(Z)|}{|Y|}, \tag{18}$$

where $\Gamma(Z)$ is the set of neighbors of Z in Y.

Note that the normalized matching property can be shown to be equivalent to another property called the *LYM property*; see Anderson [2, Sect. 2.3] for the definition of the LYM property and a proof.

Definition 4.8. Let P be a ranked poset. We say that P has the *strong matching property* if, for any levels L_1, L_2 in P such that $|L_1| \leq |L_2|$, there exists a matching from L_1 to a subset of L_2. We say that P has the *strong Sperner property* if, for any $k \geq 1$ and any k-family R in P (that is, any $R \subseteq P$ such that R contains no chains of size $k+1$), $|R|$ is no greater than the sums of the sizes of the k largest levels of P. Finally, we say that P has the *Stanley chain property* (also called *chain property* T in Griggs [11]) if, for any level L in P, there exists a set of $|L|$ disjoint chains in P such that each chain meets every level of size at least $|L|$.

The properties in Definitions 4.7 and 4.8 are related in the following way.

Lemma 4.9. *Let P be a graded poset. If P has the normalized matching property, then P has the strong Sperner property, the Stanley chain property, and the strong matching property.*

Proof. Since P has the normalized matching property, it has the strong Sperner property (see Anderson [2, Sect. 2.3]). Since P is graded and has the strong Sperner property, it also has the Stanley chain property (see Griggs [11, Thm. 1]). Finally, since P has the Stanley chain property, it also has the strong matching property (see Griggs [11, Prop. 3]). ∎

Now, to prove the Main Theorem (Section 6), we will only need to show that $\mathbf{T}_j(n)$ has the strong matching property. However, to demonstrate the extent to which $\mathbf{T}_j(n)$ resembles the full Boolean lattice, we will show that it has the normalized matching property (and therefore, all of the other matching properties mentioned above). We first need a few more definitions.

Definition 4.10. Recall that a sequence $\{a_k\}$ of nonnegative numbers is said to be *log-concave* if, for all k, we have $a_k^2 \geq a_{k+1}a_{k-1}$. (For example, a straightforward application of the binomial theorem shows that the rank numbers of $\mathbf{2}^{[n]}$ are log-concave as a function of the level.) Note that since

a_k/a_{k-1} is a nonincreasing function of k over any nonzero portion of a log-concave sequence $\{a_k\}$, a log-concave sequence with no internal zeros is unimodal; indeed, log-concavity can be thought of as a strong version of unimodality.

A ranked poset is said to be *log-concave normalized matching* if it has the normalized matching property and its rank numbers are log-concave as a function of the level.

We also need the following result of Harper [13] and Hsieh and Kleitman [14]. (See also Engel [5, Sect. 4.6].)

Lemma 4.11. *Let P and Q be log-concave normalized matching posets. Then $P \times Q$ is also log-concave normalized matching.* ▌

Corollary 4.12. *For $n, j \geq 0$, $\mathbf{T}_j(n)$ is log-concave normalized matching.*

Proof. Fixing $j \geq 0$, we proceed by induction on n. For $n = 0$, $\mathbf{T}_j(n)$ has exactly 1 element, and the corollary is clear. For $n \geq 1$, $\mathbf{T}_j(n)$ is isomorphic to a (possibly trivial) truncation of the product of $\mathbf{T}_j(n-1)$ and the chain of size 2 (Corollary 4.6). However, since the product of two log-concave normalized matching posets is log-concave normalized matching (Lemma 4.11), and since a truncation of a log-concave normalized matching poset is clearly log-concave normalized matching, the corollary follows by induction on n. ▌

Combining Corollary 4.4, Corollary 4.12, and Lemma 4.9, we have:

Corollary 4.13. *For $n, j \geq 0$, the poset $\mathbf{T}_j(n)$ has the strong matching property, the strong Sperner property, and the Stanley chain property.* ▌

Remark 4.14. For other properties implied by the fact that $\mathbf{T}_j(n)$ has the normalized matching property, see Griggs [11]. See also Question 7.1.

Remark 4.15. We remark that, using the description of the CSCD due to Greene and Kleitman [7] (see Anderson [2, Chap. 3]), it is not hard to show that the composition of the map $\{1, \ldots, n\} \to \{n, \ldots, 1\}$ (lexicographic reverse) and the map sending a subset of $[n]$ to its complement preserves the chains in the CSCD setwise, while reversing the partial order. It follows that all of our results about the tails of chains in the CSCD imply analogous results about heads of the chains in the CSCD. We leave the details to the interested reader.

5. THE INFLECTION LEVEL

Our last task before returning to the Main Theorem is to characterize the sizes of the $t_{h,k}(n)$'s in terms of the following notions.

Definition 5.1. For $n \geq 0$, we define $\lambda(n) = \left\lceil \frac{n}{2} - \frac{\sqrt{n+2}}{2} \right\rceil$ to be the *inflection level* of $\mathbf{2}^{[n]}$. The *inflection column* of $\mathbf{2}^{[n]}$ is the union of all $t_{h,k}(n)$ such that $k - h = \lambda(n)$, or in other words, the union of all CSCD chains whose tails are in $t_{0,\lambda(n)}(n)$.

Since $|t_{h,k}(n)| = \delta(n, k-h, k-h-1)$ when $t_{h,k}(n)$ is nonempty, the following theorem explains the relationship between $\lambda(n)$ and the sizes of the $t_{h,k}(n)$'s.

Theorem 5.2. *For fixed $n \geq 0$, the smallest possible r that maximizes the value of $\delta(n, r, r-1)$ in the range $0 \leq r \leq \lfloor n/2 \rfloor$ is precisely $r = \lambda(n)$.*

Proof. First, since the cases $n = 0, 1, 2$ are easily checked, we may assume that $n \geq 3$. In that case, since $\delta(n, 0, -1) = 1$ and $\delta(n, 1, 0) = n-1$, the maximum value of $\delta(n, r, r-1)$ cannot occur at $r = 0$. Therefore, assuming $r \geq 1$, we see that $\delta(n, r+1, r) - \delta(n, r, r-1)$ has the same sign as

$$\begin{aligned} D_r &= \frac{\delta(n, r+1, r) - \delta(n, r, r-1)}{n!} \\ &= \frac{1}{(r+1)!(n-r-1)!} - \frac{1}{r!(n-r)!} - \frac{1}{r!(n-r)!} + \frac{1}{(r-1)!(n-r+1)!} \\ &= \frac{(n-r+1)(n-r) - 2(r+1)(n-r+1) + (r+1)r}{(r+1)!(n-r+1)!} \\ &= \frac{n^2 - 4nr + 4r^2 - n - 2}{(r+1)!(n-r+1)!}. \end{aligned} \tag{19}$$

It follows that $D_r \leq 0$ if and only if

$$0 \geq n^2 - 4nr + 4r^2 - n - 2 = 4\left(\frac{n}{2} - r\right)^2 - n - 2, \tag{20}$$

or in other words, if and only if $r \geq \frac{n}{2} - \frac{\sqrt{n+2}}{2}$. Therefore,

$$\begin{aligned} \delta(n, r+1, r) &\leq \delta(n, r, r-1) \qquad \text{for } r \geq \frac{n}{2} - \frac{\sqrt{n+2}}{2}, \\ \delta(n, r+1, r) &> \delta(n, r, r-1) \qquad \text{for } r < \frac{n}{2} - \frac{\sqrt{n+2}}{2}, \end{aligned} \tag{21}$$

and $\delta(n, r, r-1)$ achieves its maximum value for $r = \left\lceil \frac{n}{2} - \frac{\sqrt{n+2}}{2} \right\rceil$, but not for any smaller integers. The theorem follows. ∎

Remark 5.3. Recall that the level distribution of $\mathbf{2}^{[n]}$ resembles a Gaussian distribution for large n (Figure 1 of the introduction). In terms of this comparison, the reason we call $\lambda(n)$ the inflection level of $\mathbf{2}^{[n]}$ is that $\delta(n, r, r-1)$ corresponds roughly to the first derivative of the level distribution of $\mathbf{2}^{[n]}$, and maximizing $\delta(n, r, r-1)$ corresponds roughly to an inflection point.

6. PROOF OF THE MAIN THEOREM

Before proving the Main Theorem, we obtain a refinement (Corollary 6.2) of the matching properties of $\mathbf{T}_j(n)$, using the following result.

Lemma 6.1. *Let $G(X, Y)$ be a bipartite graph. If M is any matching (vertex-disjoint set of edges) in G, then there is a matching M' of maximum cardinality (among all possible matchings) that covers all vertices covered by M.*

Proof. This is Property 5.1.5 in Asratian, Denley, and Häggkvist [3]. ▮

Corollary 6.2. *For $k \leq \lfloor n/2 \rfloor$, if $|T_{j,k}(n)| \leq |T_{j,k-1}(n)|$, then there exists a matching from $T_{j,k}(n)$ to a subset S of $T_{j,k-1}(n)$ such that $T_{j-1,k-1}(n) \subseteq S$.*

Proof. On the one hand, since $\mathbf{T}_j(n)$ has the strong matching property (Corollary 4.13), there exists a matching of cardinality $|T_{j,k}(n)|$ between $T_{j,k}(n)$ and $T_{j,k-1}(n)$. Therefore, any matching of maximum cardinality will cover all of $T_{j,k}(n)$. On the other hand, since $k \leq \lfloor n/2 \rfloor$, the chains in the CSCD provide a matching from $T_{j-1,k-1}(n)$ to $T_{j,k}(n) - t_{0,k}(n)$. Therefore, by Lemma 6.1, there exists a matching of maximum cardinality that covers $T_{j-1,k-1}(n)$, as desired. ▮

We may now prove the Main Theorem, in the form of Theorem 6.3. Let

$$d(n) = \lfloor n/2 \rfloor - \lambda(n), \tag{22}$$

$$\epsilon(n) = n - 2\lfloor n/2 \rfloor + 1 = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd,} \end{cases} \tag{23}$$

where $\lambda(n)$ is the inflection level (Definition 5.1).

Theorem 6.3. *For $n \geq 0$, we may partition $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains of size at least $d(n) + \epsilon(n)$. Furthermore, we may partition $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ skipless chains of size at least $d(n) + \epsilon(n) - 1$.*

Consider Figure 7, which shows the portion of $\mathbf{2}^{[n]}$ to the right of the inflection column (Definition 5.1), decomposed into $t_{h,k}(n)$'s. (The shaded column is the inflection column.) The idea of the proof is to take the CSCD of $\mathbf{2}^{[n]}$ and partition the indicated portion into chains with large minimum size, since the rest of $\mathbf{2}^{[n]}$ is already in chains of large minimum size. We begin by "hanging" chains resembling the chains in Figure 7 down from the middle level $\lfloor n/2 \rfloor$. (Note that in Figure 7, the shortest chains hanging from a given $t_{h,k}(n)$ in the middle level are precisely those chains whose edges all move one column to the left as they descend.) We then join these new chains to the upper halves of the CSCD chains going through the indicated portion of the middle level. Since the new chains are guaranteed to be long precisely when the old CSCD chains are short, the combined chains all have the desired minimum size. (Compare Figure 1 from the introduction.)

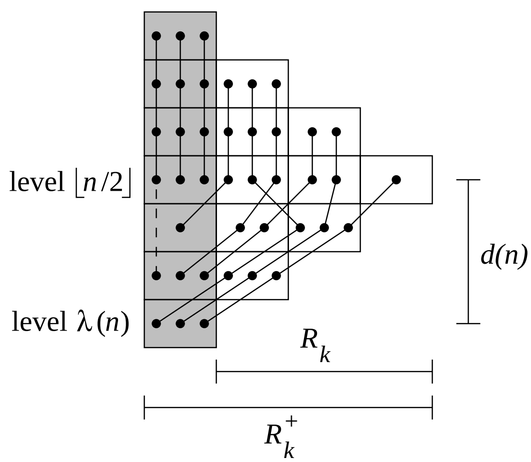

FIG. 7. Proof of the Main Theorem, $d(n) = 3$

More precisely:

Proof (Theorem 6.3). We fix n, and as shown in Figure 7, we let

$$\begin{aligned} R_k &= T_{k-\lambda(n)-1,k}(n) \\ &= \text{the portion of level } k \text{ strictly to the right of the inflection column,} \\ R_k^+ &= T_{k-\lambda(n),k}(n) \\ &= \text{the portion of level } k \text{ to the right of, or in, the inflection column.} \end{aligned}$$

Now, for $\lambda(n) < k \leq \lfloor n/2 \rfloor$, Lemma 3.5, part (3.) implies that

$$\begin{aligned} \left|R_{k-1}^{+}\right| - |R_k| &= \left|T_{k-\lambda(n)-1,k-1}(n)\right| - \left|T_{k-\lambda(n)-1,k}(n)\right| \\ &= \delta(n, k-1, \lambda(n)-1) - \delta(n, k, \lambda(n)) \\ &= \delta(n, \lambda(n), \lambda(n)-1) - \delta(n, k, k-1), \end{aligned} \tag{24}$$

which is nonnegative, since $k > \lambda(n)$ (Theorem 5.2). By Corollary 6.2, it follows that for $\lambda(n) < k \leq \lfloor n/2 \rfloor$, there exists a matching from R_k to a subset of R_{k-1}^{+} containing R_{k-1}.

Taking the union of these matchings over $\lambda(n) < k \leq \lfloor n/2 \rfloor$, we obtain a collection $\mathbf{C}_0$ of disjoint chains with the following properties:

1. For $\lambda(n) < k \leq \lfloor n/2 \rfloor$, every element of R_k is contained in a chain of $\mathbf{C}_0$.
2. For $\lambda(n) < k < \lfloor n/2 \rfloor$, every element of R_k matches both up to level $k+1$ and down to level $k-1$. It follows that the set of heads of chains in $\mathbf{C}_0$ is precisely $R_{\lfloor n/2 \rfloor}$, and that all chains in $\mathbf{C}_0$ are skipless.
3. For $\lambda(n) < k \leq \lfloor n/2 \rfloor$, every element of R_k matches down to level $k-1$, so no element of R_k is the tail of a chain in $\mathbf{C}_0$. In other words, if A is contained in a chain of $\mathbf{C}_0$, then A is the tail of its chain if and only if A is in the inflection column.
4. We claim that for $\lambda(n) \leq k \leq \lfloor n/2 \rfloor$, if A is in level k, has tail-height h, and is contained in the chain C of $\mathbf{C}_0$, then C contains at least $k - \lambda(n) - h + 1$ elements below A (inclusive). In particular, if C is a chain in $\mathbf{C}_0$ whose head has tail-height h, then

$$|C| \geq \lfloor n/2 \rfloor - \lambda(n) - h + 1 = d(n) - h + 1, \tag{25}$$

since the head of C is in level $\lfloor n/2 \rfloor$.

Now, since the claim is obvious for $k - h = \lambda(n)$, and in particular, for $k = \lambda(n)$, proceeding by induction on k, we may assume that $k > \lambda(n)$ and A is not in the inflection column. Then, from property (3.), above, there exists some node A^- in both C and the shadow of A. However, since A^- is in level $k-1$ and has tail-height at most h (Corollary 4.3), by induction, C contains at least $k - 1 - \lambda(n) - h + 1$ elements below A^-, which means that C contains at least $k - \lambda(n) - h + 1$ elements below A. The claim follows.

So now, for $A \in R_{\lfloor n/2 \rfloor}$, let $C(A)$ be the chain formed from the union of the chain of $\mathbf{C}_0$ containing A and the elements above A in its CSCD chain, and let $\mathbf{C}_1$ be the collection of all such $C(A)$. From the properties of $\mathbf{C}_0$ and the CSCD, we see that $\mathbf{C}_1$ has the following properties:

1. For $\lambda(n) < k < n - \lambda(n)$, every element of R_k is contained in a chain of $\mathbf{C}_1$. In other words, every node of $\mathbf{2}^{[n]}$ strictly to the right of the inflection column is contained in a chain of $\mathbf{C}_1$.

2. Let A be an element of $R_{\lfloor n/2 \rfloor}$ with tail-height h, and let C be the chain of $\mathbf{C}_1$ containing A. Since the head of the CSCD chain containing A is at level $n - \lfloor n/2 \rfloor + h$ (Lemma 3.3), we see that

$$|C| \geq (n - \lfloor n/2 \rfloor + h) - \lfloor n/2 \rfloor + d(n) - h + 1 = d(n) + \epsilon(n). \tag{26}$$

Next, for every CSCD chain C in the inflection column, let C^- be the nodes of C not contained in a chain of $\mathbf{C}_1$. Note that since the only nodes in the inflection column that are contained in chains of $\mathbf{C}_1$ are at level at most $\lfloor n/2 \rfloor - 1$,

$$\left|C^-\right| \geq (n - \lambda(n)) - \lfloor n/2 \rfloor + 1 = \epsilon(n) + d(n). \tag{27}$$

Let $\mathbf{C}_2$ be the collection of all such C^-. Finally, let $\mathbf{C}_3$ be the collection of all CSCD chains strictly to the left of the inflection column, and note that every chain in $\mathbf{C}_3$ has size at least $n - 2\lambda(n) + 3 = \epsilon(n) + 2d(n) + 2$. Then:

1. $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ partition $\mathbf{2}^{[n]}$ into disjoint chains;
2. Every chain in $\mathbf{C}_1$, $\mathbf{C}_2$, or $\mathbf{C}_3$ contains an element in level $\lfloor n/2 \rfloor$; and
3. Every chain in $\mathbf{C}_1$, $\mathbf{C}_2$, or $\mathbf{C}_3$ has size at least $d(n) + \epsilon(n)$.

The first statement of the theorem follows.

As for the second statement, for every $C \in \mathbf{C}_1$, let C' be the portion of C strictly to the right of the inflection column, and note that $|C'| \geq d(n) + \epsilon(n) - 1$. Then, if we let $\mathbf{C}'_1$ be the collection of all such C' and let $\mathbf{C}'_2$ be the collection of all CSCD chains to the left of the inflection column, inclusive, $\mathbf{C}'_1$ and $\mathbf{C}'_2$ partition $\mathbf{2}^{[n]}$ into the disjoint union of $\binom{n}{\lfloor n/2 \rfloor}$ skipless chains, all of size at least $d(n) + \epsilon(n) - 1$. The theorem follows. ∎

Remark 6.4. Note that for large n, in the chain partitions constructed in the proof of Theorem 6.3, roughly $\binom{n}{\lceil n/2 - \sqrt{n}/2 \rceil}$ chains are left unchanged from the CSCD of $\mathbf{2}^{[n]}$. In other words, since Stirling's approximation implies that $\binom{n}{\lceil n/2 - \sqrt{n}/2 \rceil} \Big/ \binom{n}{\lfloor n/2 \rfloor} \approx e^{-1/2} \approx 60.65\%$ for large n, roughly 40% of the chains in the partition are chains we have constructed, and roughly 60% of the chains are just taken from the CSCD.

7. OPEN QUESTIONS

We conclude with some open questions.

Question 7.1. Recall that two chains C_1 and C_2 in a ranked poset P are said to be *nested* if $|C_1| \leq |C_2|$ implies that the levels occuring in C_1 are a subset of the levels occuring in C_2, and that a *nested chain decomposition* of P is a decomposition of P into pairwise nested chains. Note that if the rank numbers of P are symmetric and unimodal, then a nested chain decomposition of P is precisely a symmetric chain decomposition of P.

We conjecture:

Conjecture 7.2. *There exists a nested chain decomposition of* $\mathbf{T}_j(n)$.

Note that a ranked poset with a nested chain decomposition has all of the matching properties in Corollary 4.13. Conversely, a ranked poset with the normalized matching property whose rank numbers are also symmetric and unimodal has a symmetric chain decomposition (a result of Anderson [1] and Griggs [8]), and it has been conjectured that every ranked poset with the normalized matching property has a nested chain decomposition (Griggs [11]). However, little progress has been made towards proving that the normalized matching property implies the existence of a nested chain decomposition, even in the case where the rank numbers are unimodal (but not symmetric), so the best approach to Conjecture 7.2 might be to take advantage of the special features of $\mathbf{T}_j(n)$. Indeed, perhaps the best approach is to construct a nested chain decomposition explicitly.

Question 7.3. It is not hard to see that the matching results of Section 4 allow us to achieve many chain decompositions of $\mathbf{2}^{[n]}$ other than the one obtained in the Main Theorem. Can these partitions be characterized in some succinct fashion? For example, thinking in terms of the majorization order on partitions (as described in the introduction), can we obtain any partition that lies between the SCD partition and the partition obtained in the Main Theorem?

Question 7.4. Note that by a detailed analysis of the posets $\mathbf{T}_j(n)$, we have obtained our results, including many matching results, without using the Kruskal-Katona Theorem (see Anderson [2, Ch. 7]), which is one of the principal methods for obtaining matching results in the Boolean lattice. (Compare Lonc [15].) Can our results be improved by the use of Kruskal-Katona?

Question 7.5. While this paper has made progress towards answering Füredi's question, there still remain significant obstacles in the way of a full answer. First of all, the chain decomposition in the Main Theorem has minimum sizes on the order of $\frac{1}{2}\sqrt{n}$, not the $\sqrt{\frac{\pi}{2}}\sqrt{n}$ required for the Füredi partition. More notably, the chain decomposition in the Main Theorem leaves all chains in the CSCD of size greater than (roughly) $\sqrt{n}$ completely unaltered; in particular, there will be many chains of size much greater than $\frac{1}{2}\sqrt{n}$ (see Remark 6.4). To achieve the Füredi partition, one would need some way of shortening these long chains, and not just lengthening the short chains, as we do here. Towards this end, can Corollary 4.13, or more speculatively, Conjecture 7.2, be used to lengthen the short chains of the CSCD with the tops and bottoms of the long chains?

REFERENCES

1. I. Anderson, *Some problems in combinatorial number theory*, Ph.D. thesis, Univ. of Nottingham, 1967.
2. ———, *Combinatorics of finite sets*, Oxford Univ. Press, 1987.
3. A. S. Asratian, T. M. J. Denley, and R. Häggkvist, *Bipartite graphs and their applications*, Cambridge Univ. Press, 1998.
4. N. G. de Bruijn, C. Tengbergen, and D. Kruyswijk, *On the set of divisors of a number*, Nieuw Arch. Wiskd. **23** (1951), 191–193.
5. K. Engel, *Sperner theory*, Cambridge Univ. Press, 1997.
6. Z. Füredi, *Problem session*, Kombinatorik geordneter Mengen (Oberwolfach, B. R. D.), 1985.
7. C. Greene and D. J. Kleitman, *The structure of Sperner k-families*, J. Comb. Thy. (A) **20** (1976), 80–88.
8. J. R. Griggs, *Sufficient conditions for a symmetric chain order*, SIAM J. Appl. Math. **32** (1977), 807–809.
9. ———, *Problems on chain partitions*, Disc. Math. **72** (1988), 157–162.
10. ———, *Saturated chains of subsets and a random walk*, J. Comb. Thy. (A) **47** (1988), 262–283.
11. ———, *Matchings, cutsets, and chain partitions in graded posets*, Disc. Math. **144** (1995), 33–46.
12. J. R. Griggs, C. M. Grinstead, and R. K. C. Yeh, *Partitioning Boolean lattices into chains of subsets*, Order **4** (1987), 65–67.
13. L. H. Harper, *The morphology of partially ordered sets*, J. Comb. Thy. (A) **17** (1974), 44–58.
14. W. N. Hsieh and D. J. Kleitman, *Normalized matching in direct products of partial orders*, Stud. Appl. Math. **52** (1973), 285–289.
15. Z. Lonc, *Proof of a conjecture on partitions of a Boolean lattice*, Order **8** (1991), 17–27.
16. B. Sands, *Problem session*, Colloquium on Ordered Sets (Szeged, Hungary), 1985.