THE GENERALIZED FÜREDI CONJECTURE HOLDS FOR FINITE LINEAR LATTICES

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Abstract. Let P be a finite rank-unimodal, rank-symmetric, normalized matching poset of width w. Generalizing a question of Füredi about the Boolean lattice, we conjecture that P has a partition into w chains such that the sizes of the chains are one of two consecutive integers. We prove this conjecture for all posets with rank ≤ 3, and consequently obtain the conjecture in the case where P has rapidly decreasing rank numbers. (A unimodal poset P has rapidly decreasing rank numbers if above (resp. below) the largest ranks of P, the size of each level is at most half of the previous (resp. succeeding) one.) The main tool for these proofs is to create new posets by judiciously collapsing some levels of P and realizing that the resulting poset continues to be normalized matching, rank-unimodal, and rank-symmetric.

One concrete corollary of our results is that there exists a partition of the linear lattices $L_n(q)$ (subspaces of an n-dimensional vector space over a finite field, ordered by inclusion) into chains such that the number of chains is the width of $L_n(q)$ and the sizes of the chains are one of two consecutive integers.

1. Introduction

The *linear lattice* or the *subspace lattice* $L_n(q)$ is the poset of all subspaces of an n-dimensional vector space over a field of q elements, ordered by inclusion. A collection, $V_0, \ldots, V_k$, of elements of $L_n(q)$ with $V_0 \subset \cdots \subset V_k$ is called a *chain* or *flag* of size $k+1$.

One of the objectives of this paper is to prove:

Main Corollary. *Let w be the number of subspaces of dimension $\lfloor n/2 \rfloor$ in $L_n(q)$. There exists a partition of $L_n(q)$ into w chains such that the sizes of the chains will be one of two consecutive integers.*

Note that the number of subspaces of dimension $\lfloor n/2 \rfloor$ in $L_n(q)$ is the *minimum* number of chains needed for a partition. The fact that this many chains are needed is immediate since no two subspaces of the same dimension can be in the same chain. That this number of chains suffice is also well known (see van Lint and Wilson [26, Theorem 24.1] or the paragraph following Theorem 2.4 below). The partition given by our theorem, in addition to having the minimum required number of chains, gives chains whose lengths are as uniform as possible.

In fact, we prove a more general result identifying a class of posets - that includes $L_n(q)$ - which can be partitioned into the minimum number of chains in such a way that the sizes of the chains are as uniform as possible.

Date: Version of October 31, 2003.

2000 *Mathematics Subject Classification.* Primary 05D05; Secondary 06A07, 06B05, 05D15.

Key words and phrases. Linear lattices, chain decompositions, generalized Füredi conjecture, normalized matching property, LYM property.

Hsu was supported by a San José State University Summer Faculty Fellowship.

Let P be a finite ranked poset, with its rank numbers (Whitney numbers) denoted by W_r, and lowest and highest ranks 0 and n, respectively. We say that P is *unimodal* if W_r is a unimodal function, and we define the *lowest mode level* (resp. *highest mode level*) of P to be the lowest (resp. highest) level r such that $W_s \leq W_r$ for all levels s. We say that P is *rank-symmetric* if $W_r = W_{n-r}$ for $0 \leq r \leq n$. We say that P is *normalized matching* (or has the *LYM property*) if, for any levels X and Y in P and $Z \subseteq X$, we have

$$\frac{|\Gamma(Z)|}{|Y|} \geq \frac{|Z|}{|X|}, \tag{1}$$

where $\Gamma(Z)$ is the set of neighbors of Z in Y.

A *chain* of size k (or length $k-1$) in P is a set of elements $u_1, \ldots, u_k$, with $u_1 < u_2 < \cdots < u_k$. This chain is *skipless* (or *saturated*) if, for $2 \leq i \leq k$, u_i covers u_{i-1}. This chain is *symmetric* if it is skipless and if $\text{rank}(u_1) + \text{rank}(u_k) = n$. P is a *symmetric chain order* if P can be partitioned into symmetric chains, and, in that case, a partition of P into symmetric chains is called a *symmetric chain decomposition* (SCD) of P. Anderson [1] and Griggs [12] independently proved that a rank-symmetric, unimodal, normalized matching poset is a symmetric chain order (see Anderson [2, Thm. 3.6.4]). Motivated by this result we say that a poset is *Anderson-Griggs* if it is rank-symmetric, unimodal, and normalized matching. A set of incomparable elements in a poset is called an *anti-chain* and the maximum size of an anti-chain is the *width* of the poset. By Dilworth's celebrated Theorem [5] (or see Theorem 3.2.1 of Anderson [2]), the width of a poset is the same as the minimum number of chains needed to partition the poset into chains.

Our main conjecture is a generalization of a question that Füredi [8] originally asked about the Boolean lattice. Recall that $\mathbf{2}^{[n]}$, the *Boolean lattice* or *subset lattice* of order n, is the poset of subsets of $[n] = \{1, \ldots, n\}$ ordered by inclusion. It is well known (see Anderson [2]) that $\mathbf{2}^{[n]}$ is an Anderson-Griggs poset.

Generalized Füredi Conjecture. *Let P be an Anderson-Griggs poset of width w. Then we can partition P into w chains such that the sizes of the chains are one of two consecutive integers.*

To state our main theorem we need one more definition. Let P be a unimodal ranked poset with rank numbers W_r and lowest and highest mode levels $R_0 \leq R_1$. We say that P has the *rapidly decreasing rank numbers* property, or *RDR property*, if for $r \leq R_0$, we have $W_r/W_{r-1} \geq 2$, and for $r \geq R_1$, we have $W_r/W_{r+1} \geq 2$.

Main Theorem. *The generalized Füredi conjecture holds for any Anderson-Griggs poset with the RDR property.*

Our main corollary follows from our main theorem by noticing that finite linear lattices are Anderson-Griggs posets with the RDR property. We prove the above by first proving our main lemma which provides further evidence for the generalized Füredi conjecture:

Main Lemma. *The generalized Füredi conjecture is true for any Anderson-Griggs poset with three levels.*

The subject of chain partitions of posets in general and the Boolean lattice in particular has been the object of much research. We start our short review of the story with Sands [25] who asked if, for a given k and for large enough n, $\mathbf{2}^{[n]}$ can be partitioned into chains of size 2^k. The answer is clearly yes for $k = 0, 1$ and Griggs, Grinstead, and Yeh [15] proved that $\mathbf{2}^{[n]}$ can be partitioned into chains of size 4 if and only if $n \geq 9$. Griggs [13] later conjectured that for a given $c \geq 1$ and for n sufficiently large, $\mathbf{2}^{[n]}$ can be partitioned into chains of size c and a single chain of size at most $c - 1$. Griggs [14] himself modified the standard inductive construction of a symmetric chain decomposition of $\mathbf{2}^{[n]}$ (see Anderson [2, Chapter 3] or van Lint and Wilson [26, page 55]) to create a chain decomposition of $\mathbf{2}^{[n]}$ with a large number of chains of size at most c. In particular, applying this construction, and thinking of the desired chain size c as a function of n, if $c(n) = o(\sqrt{n})$, the proportion of subsets in $[n]$ that belong to chains of size c approaches 1 as $n \to \infty$, and if $c(n) \sim a\sqrt{n}$ for some constant a, the same proportion approaches a constant that depends on a and is strictly less than 1. The most significant progress in this direction was made by Lonc [19], who proved this conjecture of Griggs hence settling both Griggs' conjecture and the original question by Sands. Note, however, that in Lonc's proof, n is required to be quite large, as a function of c; in fact, for a given c, it is not hard to see that $2^{\exp(c^2)}$ is a (coarse) lower bound for the required size of n in Lonc's proof. In other words, for a given n, only a relatively small desired chain size c can be achieved. Recently, Elzobi and Lonc [6] improved this bound slightly and proved that, for any n, as long as $c \leq \frac{1}{6} \left\lfloor \sqrt{\log \log n} \right\rfloor$, one can partition $\mathbf{2}^{[n]}$ into chains of size c and a single chain of size at most $c - 1$.

As was mentioned before, the minimum number of chains needed to partition a poset is the same as its width. By focusing on partitions of a poset into the minimum number of chains we ensure that we do not end up with a large number of very short chains. Taking this approach Füredi [8] asked if $\mathbf{2}^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains such that the size of every chain is one of two consecutive integers. Such a partition would have the minimum number of chains as well as uniform chain sizes. For example, the Füredi partition of $\mathbf{2}^{[10]}$ would require 16 chains of size five and 236 chains of size 4. Our "Generalized Füredi Conjecture" generalizes this question and the current paper proves the generalization for Anderson-Griggs posets with the RDR property in general, and for the finite linear lattices in particular.

Füredi's question for the Boolean lattice certainly remains open. The authors (together with Christopher Towse) recently made progress in two papers [16, 17]. By Stirling's approximiation, Füredi's conjecture asks for a partition of $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains of size asymptotic with $\sqrt{\frac{\pi}{2}}\sqrt{n}$. In [16], we constructed a chain decomposition of $\mathbf{2}^{[n]}$ with the desired number of chains and with all chain sizes more than roughly $\frac{1}{2}\sqrt{n}$. In a second paper [17], we construct a chain partition of $\mathbf{2}^{[n]}$ for $n > 16$ that, in addition to having the right number of chains and the same estimate for the lower bound of the sizes of the chains, has no chain longer than $\frac{3}{2}\sqrt{n \log n}$. More precisely:

Theorem 1.1. *For any $c > 1$, there exist functions $e(n) \sim \frac{1}{2}\sqrt{n}$ and $f(n) \sim c\sqrt{n \log n}$ and an integer N (depending only on c) such that for all $n > N$, there is a decomposition of the Boolean lattice $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains, all of which have size between $e(n)$ and $f(n)$.*

In fact, the integer N is really only needed for the upper bound, as the construction in [16] gives a partition of $\mathbf{2}^{[n]}$ into $\binom{n}{\lfloor n/2 \rfloor}$ chains such that the size of the shortest chain is approximately $\frac{1}{2}\sqrt{n}$ for *every* n. Even when we consider the upper bound, N is not "too large"; for example, if $c = 1.5$, then $N = 16$ suffices.

The most general question about chain partitions of the Boolean lattice is due to Griggs [13], who asked: Given a partition $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_\ell)$ of 2^n into positive parts, is there a partition of $\mathbf{2}^{[n]}$ into chains of sizes $\mu_1, \ldots, \mu_\ell$? In fact, Griggs also conjectured an answer to this question, the statement of which requires the following terminology. A recursive construction of a symmetric chain decomposition for the Boolean lattice was given by deBruijn, Tengbergen, and Kruyswijk [4] (or see Greene and Kleitman [11, page 30], or Anderson [2, chapter 3]), and it is also straightforward to see that if $\sigma_k = n - 2j + 1$ for all k and j such that $\binom{n}{j-1} < k \leq \binom{n}{j}$ and $0 \leq j \leq \lfloor n/2 \rfloor$, then the partition

$$\boldsymbol{\sigma} = (\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\binom{n}{\lfloor n/2 \rfloor}}) \tag{2}$$

of 2^n, which we call the *SCD partition*, has the property that the chains in any symmetric chain decomposition of $\mathbf{2}^{[n]}$ have sizes σ_1, σ_2, and so on. Finally, recall that the *dominance* (or *majorization*) order on partitions of an integer m is defined by the rule that for partitions $\boldsymbol{\mu} = (\mu_i)$ and $\boldsymbol{\nu} = (\nu_i)$ of m, we have that $\boldsymbol{\mu} \leq \boldsymbol{\nu}$ if and only if for all j,

$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \nu_i. \tag{3}$$

Griggs' conjecture [13] is that for a partition $\boldsymbol{\mu}$ of the integer 2^n, there exists a partition of $\mathbf{2}^{[n]}$ into chains with sizes given by $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu} \leq \boldsymbol{\sigma}$ in the dominance order.

The Füredi question remains the most difficult case of the Griggs conjecture and yet not much work, beyond what was reported above, has been done on this conjecture of Griggs. We offer a generalization of the Griggs conjecture as follows:

Generalized Griggs Conjecture. *Let P be any Anderson-Griggs poset. Let $\boldsymbol{\sigma}$ be the partition of the integer $|P|$ whose parts are the chain sizes in a symmetric chain decomposition of P. Let $\boldsymbol{\mu}$ be an arbitrary partition of the integer $|P|$ into positive parts. Then it is possible to find a chain partition of P with chain sizes equal to the parts of $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu} \leq \boldsymbol{\sigma}$ in the dominance order.*

The only case of the generalized Griggs conjecture that has been proved is when P is the product of two chains. This is the main result of Lonc and Elzobi in [20]. While we do not prove this conjecture for any poset, our proof of the generalized Füredi conjecture for Anderson-Griggs posets with three levels, is certainly a good start for the proof in this case.

We now outline the rest of this paper. After summarizing some necessary background material (Section 2), we show that all finite linear lattices $L_n(q)$ are RDR (Section 3). We then proceed to prove the generalized Füredi conjecture for 3-level Anderson-Griggs posets (Theorem 4.2), and use it to prove the generalized Füredi conjecture for all Anderson-Griggs posets with the RDR property (Section 4), including the finite linear lattices. We conclude with further observations and some open problems (Section 5).

2. Background

In this section, we summarize some relevant background material. Standard references are Anderson [2] and Engel [7]. Greene and Kleitman [11] and van Lint and Wilson [26] provide relevant background material as well.

Central to our discussion are normalized matching posets. We begin with the following easy consequence of the Hall Marriage Theorem.

Lemma 2.1. *If P is a normalized matching poset, then for any levels L_1, L_2 in P such that $|L_1| \le |L_2|$, there exists a matching from L_1 to a subset of L_2.* □

We will also need the main tool developed in [17], the statement of which uses the following definition.

Definition 2.2. Let P be a ranked poset. A *rank-collected* poset of P is any poset P' obtained from P by repeatedly removing levels and combining consecutive levels (i.e., making consecutive levels incomparable).

Theorem 2.3 (Rank-collection)**.** *Any rank-collected poset of a normalized matching poset is normalized matching.* □

In fact, Lemma 4.1 is really just a variation on Theorem 2.3, so the reader interested in the proof of Theorem 2.3 can find the essential idea there.

Finally, we review the facts about linear lattices that we will need. We use the following standard q-binomial notation:

$$\begin{aligned} [n] &= \frac{q^n - 1}{q - 1}, \\ [n]! &= [n][n-1]\cdots[1], \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]!}{[k]![n-k]!}. \end{aligned} \tag{4}$$

Theorem 2.4. *The linear lattice $L_n(q)$ is Anderson-Griggs, and its kth rank number is given by $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. See, for example, Engel [7, Sects. 4.5, 5.1] and Goldman and Rota [9, 10]. □

Note that since every symmetric chain of $L_n(q)$ contains a single subspace of dimension $\lfloor n/2 \rfloor$, the fact that $L_n(q)$ is a symmetric chain order implies that its width (i.e., the minimum number of chains in a chain partition) equals the number of subspaces of dimension $\lfloor n/2 \rfloor$. We should also mention that one reason for the interest in $L_n(q)$ is that every finite projective lattice of rank ≥ 4 is isomorphic to some linear lattice $L_n(q)$ (see Chapter 23 and Theorem 23.6 of van Lint and Wilson [26, Chapter 23]).

3. Rapidly decreasing rank numbers

Recall that a ranked poset with rank numbers W_r is said to have the RDR property if the rank numbers are unimodal, the maximum occuring for $R_0 \le r \le$

R_1, and we have

$$W_r/W_{r-1} \geq 2 \quad \text{for } r \leq R_0$$
$$W_r/W_{r+1} \geq 2 \quad \text{for } r \geq R_0$$

Our main examples of posets with the RDR property are the linear lattices $L_n(q)$, as shown by the following theorem.

Theorem 3.1. *For any finite linear lattice $L_n(q)$, we have that*

$$\begin{bmatrix} n \\ r \end{bmatrix}_q \Big/ \begin{bmatrix} n \\ r-1 \end{bmatrix}_q \geq q^{n-2r+1}. \tag{5}$$

In particular, since $q^{n-2r+1} \geq q \geq 2$ for $r \leq n/2$, $L_n(q)$ is an RDR poset.

Proof. By (4), we see that the left-hand side of (5) is equal to

$$\frac{[n]!}{[r]![n-r]!} \cdot \frac{[r-1]![n-r+1]!}{[n]!} = \frac{[n-r+1]}{[r]} = \frac{q^{n-r+1}-1}{q^r-1}. \tag{6}$$

However, since

$$q^{n-r+1} - 1 \geq q^{n-r+1} - q^{n-2r+1} = (q^r-1)(q^{n-2r+1}), \tag{7}$$

we see that (6) is at least q^{n-2r+1}, and (5) follows. The rest of the theorem follows by the definition of RDR and the rank-symmetry of $L_n(q)$. ☐

4. The 3-level generalized Füredi conjecture

In this section, we prove our Main Lemma, that is the generalized Füredi conjecture for posets with 3 levels (Theorem 4.2) and use it to obtain our Main Theorem and Main Corollary, in the form of Theorem 4.3. First, however, we need the following lemma.

Lemma 4.1. *Let P be a normalized matching poset with exactly 3 levels, Y_1, X, and Y_2, from lowest to highest. Let $Y = Y_1 \cup Y_2$, and form a bipartite graph Γ with vertices $X \cup Y$ and an edge between $x \in X$ and $y \in Y$ if and only if either $x \leq y$ or $x \geq y$. Then if $|X| \leq |Y|$, there exists a matching from X to a subset of Y, and vice versa.*

Proof. If we can verify (1) for $Z \subseteq X$, the analogous inequality for subsets $W \subseteq Y$ follows by considering complements (see the proof in Engel [7, Prop. 4.5.2]), and the lemma follows easily from the Hall Marriage Theorem. So let Z be a subset of X, let $\Gamma(Z)$ be the set of neighbors of Z in Y, and for $i = 1, 2$, let $\Gamma_i(Z)$ be the set of neighbors of Z in Y_i. Since P is normalized matching, for $i = 1, 2$, we have:

$$|\Gamma_i(Z)| \geq |Z| \left(\frac{|Y_i|}{|X|}\right). \tag{8}$$

Therefore, since Y_1 and Y_2 are disjoint,

$$|\Gamma(Z)| = |\Gamma_1(Z)| + |\Gamma_2(Z)| \geq |Z| \left(\frac{|Y_1| + |Y_2|}{|X|}\right) = |Z| \left(\frac{|Y|}{|X|}\right). \tag{9}$$

The lemma follows. ☐

We now come to the proof of our Main Lemma which is the main technical result of this section.

Theorem 4.2. *Let P be an Anderson-Griggs poset with 3 levels. The generalized Füredi conjecture holds for P.*

Proof. Let T, M, and B be the top, middle, and bottom levels of P, respectively, let $m = |M|$, and let $k = |T| = |B|$ (by the rank-symmetry of P). Note that by unimodality, $m \geq k$.

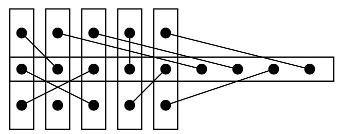

FIGURE 1. Biological and foster parents and children

First, let C be a symmetric chain decomposition of P. Such a partition exists because P is Anderson-Griggs (see Anderson [2, Theorem 3.6.4]). In this context, it will be convenient to think of the elements of M as *children* and the elements of T and B as *parents*. Note that each child $c \in M$ that is contained in a chain of size 3 in the partition C is thereby matched with two parents $t \in T$ and $b \in B$; we call these the *biological* parents of c. (Biological relationships are illustrated by the boxes in Figure 1). We call the children contained in chains of size 1 in C *orphans*.

We now have two (slightly redundant) cases: Either $m \geq 2k$, or $2k \geq m$. In the first case, by Lemma 4.1, we may match every parent with a different *foster* child, which may or not be also one of its biological children, to muddle the metaphor slightly. In the second case, this matching works the other way around, in that every child is matched with a different foster parent. Either way, we call this matching the *foster* matching. (The foster matching is illustrated by the lines in Figure 1, in the case $2k \geq m$.) In both cases, we will use some combination of the foster matching and the biological matching to obtain the Füredi partition of P.

Case 1: $m \geq 2k$. In that case, we put each parent in a chain with its foster child. Since there are more children than parents, we obtain a chain partition with a minimal number of chains in which all chains have size 1 or 2.

Case 2: $2k \geq m$. In that case, we begin with the symmetric chain partition C and modify it until C no longer has orphans, using the following procedure. Arbitrarily choose an orphan $c \in M$ in the partition C, and rearrange C by moving the foster parent p of c from its current chain to form a chain of size 2 with c. Note that p is taken from a chain containing its biological child c', which is therefore left with either 1 parent or 0 parents, the latter occurring precisely if the other biological parent of c' has been moved in some previous step. Therefore, either c' is left with 1 parent (top of Figure 2), or c' is left with 0 parents and becomes an orphan (bottom of Figure 2).

Repeatedly applying this process to the new set of orphans of C, we note that once an orphan is matched with its foster parent, it cannot be re-orphaned, since foster parents are only taken from their biological children, by the uniqueness of the foster matching. Therefore, we must eventually run out of orphans, making C a partition with all chains of size 2 or 3. The theorem follows. ☐

We are now ready to prove our Main Theorem which together with Theorem 3.1 immediately implies our Main Corollary.

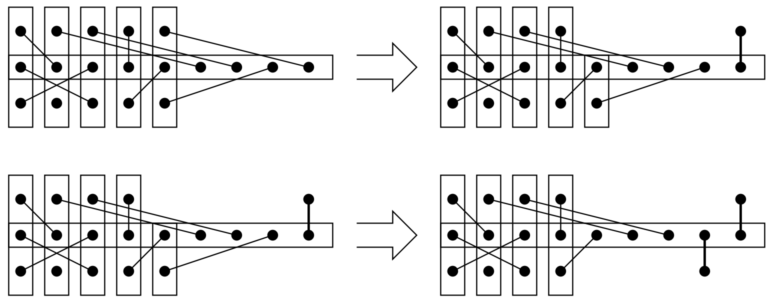

FIGURE 2. Foster matching leaves either 1 or 0 parents

Theorem 4.3. *The generalized Füredi conjecture holds for any RDR Anderson-Griggs poset.*

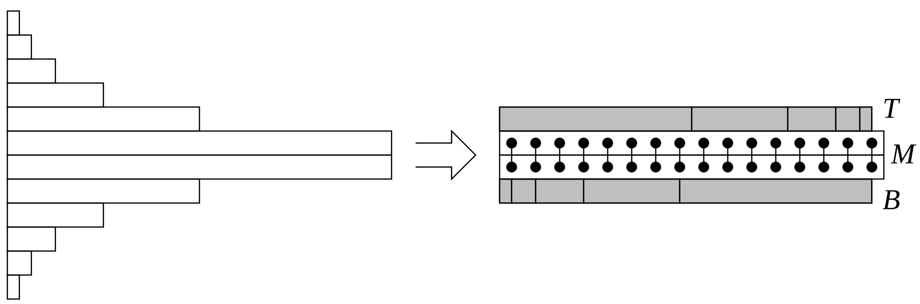

FIGURE 3. Proof of Füredi conjecture for RDR Anderson-Griggs posets

Proof. Let P be an RDR Anderson-Griggs poset. We assume P has non-mode levels, as the corollary is otherwise trivial. By the RDR property, every level below the lowest mode level is at most half the size of the level above it. From the sum of a geometric series with ratio $1/2$, we see that the sum of the sizes of all levels below the lowest mode level is less than the size of a mode level. Similarly, the sum of the sizes of all levels above the highest mode level is also less than the size of a mode level. Therefore, using Theorem 2.3 first to collect all ranks below the mode levels, and then to collect all ranks above the mode levels, and noting that this preserves rank-symmetry, we obtain an Anderson-Griggs poset P_1 with exactly one level T above the mode levels, and one level B below the mode levels, as shown in Figure 3.

It remains only to spread the top level T and the bottom level B evenly (so to speak). P_1 is an Anderson-Griggs poset and such posets have symmetric chain decompositions (Anderson [2, Theorem 3.6.4]). Choose one such SCD and call it C. We then form a 3-level poset P_2 by taking the top level to be T, the bottom level to be B, and the middle level M to be the portions of the chains of C that lie between the lowest and highest mode levels, as shown in Figure 3. The poset P_2 is clearly rank-symmetric unimodal; furthermore, since the definition of normalized matching (see the Introduction) works level-by-level, P_2 is also normalized matching. Therefore, by Theorem 4.2, P_2 has a Füredi partition, which in turn induces a Füredi partition of P. The theorem follows. □

5. Observations, open problems, and acknowledgments

We end this paper with a few remarks and a discussion of open questions.

Remark 5.1. Note that at no point in this paper have we needed to know the actual chain sizes in a Füredi partition of the lattices $L_n(q)$. For the interested reader, we discuss this question here.

Fix q, and let m_n be the lowest mode level of $L_n(q)$. By Theorem 3.1, we see that for $r \leq m_n$, the ratio between the sizes of level r and level $r-1$ is at least q, which means that the sums of the sizes of the levels strictly below level m_n is at most $1/q + 1/q^2 + 1/q^3 + \cdots = 1/(q-1)$ times the size of level m_n. Therefore, for $q \geq 3$, the sum of the sizes of all non-mode levels of $L_n(q)$ is at most the size of a mode level. It follows that for $q \geq 3$, the smaller of the two chain sizes in a Füredi partition of $L_n(q)$ is always just the number of mode levels of $L_n(q)$, that is, 1 for n even and 2 for n odd.

For $q = 2$, we note that the proof of Theorem 3.1 shows that the ratio between the sizes of level m_n and level $m_n - 1$ approaches 2 and the ratio between the sizes of level $m_n - 1$ and level $m_n - 2$ approaches 8 as $n \to \infty$. Therefore, for sufficiently large n, the sum of the sizes of all non-mode levels of $L_n(2)$ will be greater than the size of a mode level, which means that, for all such n, the smaller of the two chain sizes in a Füredi partition of $L_n(2)$ is precisely 1 greater than the number of mode levels of $L_n(2)$, that is, 2 for n even and 3 for n odd. Thus we have

Corollary 5.2. *Let w be the width of $L_n(q)$ and assume that $L_n(q)$ has been partitioned into w chains such that the sizes of the chains are one of two consecutive integers. Then*

1. *for $q \geq 3$, the sizes of chains are*
   - *1 and 2 if n is even*
   - *2 and 3 if n is odd*
2. *for $q = 2$, and large enough n, the sizes of the chains are*
   - *2 and 3 if n is even*
   - *3 and 4 if n is odd*

Remark 5.3. Note that Boolean lattices are not RDR; in fact, the Boolean analogue of (6) shows that $\binom{n}{r} \Big/ \binom{n}{r-1} \geq 2$ only for $r \leq (n+1)/3$. Therefore, if we apply the approach of Corollary 4.3 to the Boolean lattice, the corresponding results are not as good as those previously obtained in [17].

We conclude by describing some open problems.

1. The original Füredi conjecture for the Boolean lattice (see Introduction) is still definitely open; see [16, 17] for recent progress. As for the generalized Füredi conjecture, it would be interesting to see what results can be obtained for other Anderson-Griggs lattices, such as the divisor lattice and other products of Anderson-Griggs lattices with log-concave rank numbers (which are also Anderson-Griggs lattices with log-concave rank numbers; see Engel [7, Sect. 4.6]).

2. All cases of the generalized Griggs conjecture are open except the product of two chains (Lonc and Elzobi [20]). Note, however, that the proof of Theorem 4.2 gives a reasonable start to this conjecture in the case of an Anderson-Griggs poset with 3 levels. As for more substantial cases, like linear lattices or the Boolean lattice, little is known.
3. Does some analogue of the Füredi conjecture hold for linear lattices over infinite fields? Note that such lattices do have symmetric chain decompositions, a result of Vogt and Voigt [27]. It is not completely clear what the analogue of the Füredi conjecture should be, but given the results of this paper, it at least seems plausible to think that any dimension n linear lattice has a decomposition into chains of size 1 or 2 (resp. 2 or 3) for n even (resp. odd).
4. Very little is known about analogous questions for posets that are not rank-symmetric. For example, the analogue of the Anderson-Griggs theorem (that is, every Anderson-Griggs poset is a symmetric chain order) for non-rank-symmetric posets is the conjecture that any rank-unimodal normalized matching poset is *nested*, another conjecture of Griggs (see Griggs [13] for definitions and details). It would be interesting to see if it helps to consider the RDR case.

   For any nested rank-unimodal normalized matching poset, following the reasoning from the Introduction, it is reasonable to look at the direct analogues of the Griggs and Füredi conjectures. The lattice of faces of a cube (see Metropolis and Rota [23, 22] and Metropolis, Rota, Strehl, and White [24]), which is still not known to be nested, is a particularly interesting example to consider.
5. It has been attributed [21, 18] to Gian-Carlo Rota that for every theorem that holds for the Boolean lattice, there should be analogues for the linear lattices $L_n(q)$ and the *partition lattices* Π_n (the partitions of $\{1, \ldots, n\}$, ordered by refinement). However, since Π_n is not normalized matching for $n \geq 20$, and is not even Sperner for sufficiently large n (see Engel [7, Sect. 5.4]), no direct analogue of the Füredi and Griggs conjectures can hold for Π_n. It would be quite interesting to find an analogue that does.

Acknowledgments. We wish to thank the Pomona College Math Department and Kathy and Greg Sheldon for generously hosting Hsu and Logan while this research was conducted. Shahriari thanks the Institute for Studies in Theoretical Physics and Mathematics (IPM) in Tehran for its hospitality during the August 2003 International Workshop on Combinatorics, Linear Algebra, and Graph Coloring where an earlier draft of this paper was presented as a keynote talk.

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