

Some Very Interesting Sequences

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July 6, 2006

1 Introduction

First of all, we hope that anyone reading this is a Nerd. (Certainly the authors are, and quite proud of it!) Nerds have been interested in sequences of numbers throughout the ages, perhaps ever since Nerds have existed, and since we'll be discussing some of our favorite sequences and their remarkable properties, let's face it — only a real Nerd is likely to be interested.

So everyone reading this far is a Nerd, then? Great! Let's begin.

2 Fibonacci

As Nerds, you have probably already seen the *Fibonacci sequence* at some point:

$$0 \quad 1 \quad | \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \dots$$

As you may remember, the rule for the Fibonacci sequence is that the first two terms are 0 and 1, and thereafter, each term is the sum of the previous two.

Of course, there's nothing special about starting with 0 and 1; we might well start with, say, 1 and 3:

$$\begin{array}{cc|cccccc} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \dots \\ 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & \dots \end{array}$$

In fact, some of you may recognize this second sequence as the *Lucas sequence*, though the Lucas sequence usually begins 2, 1, 3, ...

What you may not know is that there is a way to choose infinitely many of these rows that has many surprising properties! To wit:

$$\begin{array}{cc|cccccc} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \dots \\ 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & \dots & \text{(you} \\ 2 & 4 & 6 & 10 & 16 & 26 & 42 & 68 & \dots & \text{fill} \\ 3 & 6 & & & & & & & & \text{in} \\ 4 & 8 & & & & & & & & \text{the} \\ 5 & 9 & & & & & & & & \text{rest} \\ 6 & 11 & & & & & & & & \text{...)} \end{array}$$

This particular choice of the initial two columns produces an infinite array, to the right of the dividing line, called the *Wythoff Array*. As first discovered by mathematician David Morrison [7] in 1980, the Wythoff Array has the wonderful property that every positive integer appears exactly once in it (again, to the right of the dividing line). Try filling in some more of the Wythoff Array to see that this works, at least in rows 0 through 6.

The only part of the rules producing the Wythoff Array that may still be unclear is how we choose the numbers in the first two columns. The first column is easy: take the obvious sequence 0, 1, 2, 3, ... As for the second column, in the row that begins with the number n , find the place in the Wythoff Array (i.e., to the right of the dividing line) where n appears, take number that appears immediately after that, and add 1. For example, since 5 appears in the top row of the Wythoff Array, and the number that comes after 5 is 8, the second number in row 5 is 9. Similarly, the number that comes after 6 in the Wythoff Array is 10, so row 6 begins 6 11 | ... Try to start the next two rows (rows 7 and 8) to see if you understand the rule. (Answers below.¹)

3 Speak, Say

Now how about this one? Here are the first few terms:

1, 11, 21, 1211, 111221, ...

Do you see the pattern yet? It's not easy; neither of us got it on the first try. How about some more terms, picking up where we left off:

..., 312211, 13112221, 1113213211, ...

Seen enough? Or perhaps we should say, have you heard enough? If you still don't hear the pattern, try saying each term (after the initial 1) out loud:

“one one”, “two one”, “one two, one one”, “one one, one two, two one”, ...

This sequence is therefore called the “Speak-and-say” sequence: Each term comes from reading the previous one out loud. (Can you figure out the next two terms we haven't supplied? Answers below.²)

It may surprise you to hear that we can actually come to some mathematically interesting conclusions about the “Speak-and-say” sequence. For example, JHC (the first author) proved that as n goes to infinity, the n th root of the n th term approaches a limit $\ell \approx 1.3\dots$, where ℓ is a root of a certain polynomial of degree 71. In fact, JHC and Richard Parker showed that *no matter how you begin the sequence*, this same limiting property holds for *any* sequence constructed by this rule — with one exception. Can you find the exception, which simply repeats the same term *ad infinitum*? (Answers below.³)

¹Rows 7 and 8 begin 7 12 | ... and 8 14 | ..., respectively.

²The terms after 1113213211 are 31131211131221 and 13211311123113112211.

³The exception is 22, 22, 22, ...

As a historical note, the limiting property (or more generally, JHC’s “Cosmological Theorem”) was first proved in the late 1970’s, as described above, but the details of that original proof, which included a long check of various cases, were lost for some time. Then in 1997, Shalosh B. Ekhad and Doron Zeilberger used a computer program to automate the case check and thus recover the proof; in fact, their program was able to recreate much of the proofs of JHC and Parker “from scratch”, that is, without having to tell the computer what the answer is beforehand. The interested reader should consult Ekhad and Zeilberger [3] for details.

4 Stern and Fractran

Suppose, for reasons we’ll reveal in a bit, we consider the following Fibonacci-ish function:

$$\begin{aligned} f(0) &= 0, & f(2n) &= f(n), \\ f(1) &= 1, & f(2n+1) &= f(n) + f(n+1) \end{aligned}$$

Write this sequence out, and then look at the fractions $\frac{f(n)}{f(n+1)}$ formed by pairs of consecutive terms:

$n :$	0	1	2	3	4	5	6	7	8	9
$f(n) :$	0	1	1	2	3	5	8	13	21	34
$\frac{f(n)}{f(n+1)} :$		$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{13}{21}$	$\frac{21}{34}$

This sequence $f(n) = 0, 1, 1, 2, 1, 3, \dots$ is called *Stern’s diatomic series*, and has the remarkable property that every nonnegative rational number (i.e., every ratio of two positive integers, along with $0=0/1$) appears *exactly once* in the sequence $\frac{f(n)}{f(n+1)}$ of fractions, a fact discovered in 1858 by the mathematician Moritz A. Stern.

Another sequence arising from fractional fun can be found in the following fourteen fractions:

A	B	C	D	E	F	G	H	I	J	K	L	M	N
$\frac{17}{91}$	$\frac{78}{85}$	$\frac{19}{51}$	$\frac{23}{38}$	$\frac{29}{33}$	$\frac{77}{29}$	$\frac{95}{23}$	$\frac{77}{19}$	$\frac{1}{17}$	$\frac{11}{13}$	$\frac{13}{11}$	$\frac{15}{14}$	$\frac{15}{2}$	$\frac{55}{1}$

The integer sequence coming from that list is made by starting with the number 2, and then repeatedly multiplying by the first fraction in the list that will produce an integer. For example, since the first fraction where the denominator divides 2 is $M = \frac{15}{2}$, the second term in the sequence (after 2) is $2M = 15$. Since none of the denominators divide 15 except the denominator of the last fraction $N = \frac{55}{1}$, the third term in the sequence is $15N = 3 \cdot 5^2 \cdot 11 = 825$.

Continuing in the same vein, the first few terms are:

$$2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770, 910, \\ 170, 156, 132, 116, 308, 364, 68, 4 = 2^2, \dots$$

We note that $4 = 2^2$ because it turns out that the only powers of 2 that appear in this sequence follow the pattern:

$$2 \dots 2^2 \dots 2^3 \dots 2^5 \dots 2^7 \dots 2^{11} \dots 2^{13} \dots$$

In other words, the only powers of 2 appearing in this sequence are the numbers 2^p , where p is a prime number, in the correct order!

So what's the secret, you ask? The answer is that those 14 fractions A through N are actually a very cleverly encoded computer program for listing the prime numbers. (For those of you familiar with the *Sieve of Eratosthenes*, the program tries to divide each positive number by all of the previous ones, and when it finds a number n that can only be divided by 1 and itself, it displays the power 2^n .) In fact, the "language" in which this program is written is called FRACTRAN, and one can even show that in theory, *any* computer program can be re-written in FRACTRAN. (This is admittedly true of most computer languages, but still, a notable fact.)

It is worth mentioning that subsequent to JHC's discovery of the above fourteen fraction program for the primes, in 1999, a doctoral student in mathematics at the University of Western Australia named Devin Kilminster (now a lecturer at Oxford University), devised a prime program in FRACTRAN using only *ten* fractions [5]:

$$\begin{array}{cccccccccc} A & B & C & D & E & F & G & H & I & J \\ \frac{7}{3} & \frac{99}{98} & \frac{13}{49} & \frac{39}{35} & \frac{36}{91} & \frac{10}{143} & \frac{49}{13} & \frac{7}{11} & \frac{1}{2} & \frac{91}{1} \end{array}$$

In Kilminster's version, we start with the number 10, and the only powers of 10 appearing in the sequence are the numbers 10^p , where p is a prime number, in the correct order.

5 Tangent, Secant, Triangles, Zigzag Permutations

Those of you who have taken calculus may have seen that:

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

for all real numbers x , where x is measured in radians. For those of you who haven't taken calculus, fret not; to convince yourself that these formulas work,

get a good calculator, make sure the calculator is in radians mode, pick your favorite relatively small value of x , like $x = 0.1$ or so, and try comparing both sides of each equation yourself. Of course, since the right-hand sides of these equations go on forever, you won't quite get what's on the left-hand side, but the farther out you go, the closer you'll get. (After all, how do you think your calculator computes $\sin x$ and $\cos x$?)

Now, the pattern in the sequence of coefficients in the "series expansions" of $\sin x$ and $\cos x$ should be fairly clear. On the other hand, the coefficient patterns in the following similar but lesser-known expansions are harder to see:

$$\begin{aligned}\tan x &= \frac{1x^1}{1!} + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \dots \\ \sec x &= \frac{1x^0}{0!} + \frac{1x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots\end{aligned}$$

Can you see a pattern in the sequence 1, 2, 16, 272, ... or the sequence 1, 1, 5, 61, ...? The answer lies in the *zigzag triangle*:

$$\begin{array}{cccccccc} & & & & & & & 1 \\ & & & & & & 0 & \rightarrow & 1 \\ & & & & & 1 & \leftarrow & 1 & \leftarrow & 0 \\ & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 \\ & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 \\ 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \rightarrow & 16 \\ 61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 & \leftarrow & 0\end{array}$$

The secant numbers, or *zig numbers*, or *Euler numbers* (see below for more on Euler), can be seen on the left-hand border of the triangle, and the tangent numbers, or *zag numbers*, are on the right-hand border of the triangle (after the first row). In fact, the secant and tangent numbers also appear as the row sums of the triangle — can you see how?⁴

The zigzag triangle is constructed by starting with 1 in the top row, starting each succeeding row with 0, alternating left-to-right rows with right-to-left rows (the "boustrophedon" rule), and then applying the zigzag addition rule

$$\begin{array}{c} a \\ b \rightarrow (a + b)\end{array}$$

in the direction indicated by the arrows. Can you figure out the entries in the next row of the zigzag triangle?⁵

Now, the numbers in the series expansions of important functions usually count something, and the tangent and secant numbers are no exception; they count *zigzag permutations*. A zigzag permutation of $1, \dots, n$ is an arrangement

⁴The tangent numbers are the sums of the rows with \leftarrow 's, and the secant numbers are the sums of the rows with \rightarrow 's.

⁵The next row is constructed from left to right, and looks like $0 \rightarrow 61 \rightarrow 122 \rightarrow 178 \rightarrow 224 \rightarrow 256 \rightarrow 272 \rightarrow 272$.

of the numbers 1 through n in a way so that the numbers alternately increase and decrease. For example, 14253 is a zigzag permutation of 1, 2, 3, 4, 5, whereas 15234 is not a zigzag permutation (why not?).

To give more examples, for $n = 1, 2, 3, 4$, the n th row of the following table lists all zigzag permutations of $1, \dots, n$, ordered by the last number in the permutation.

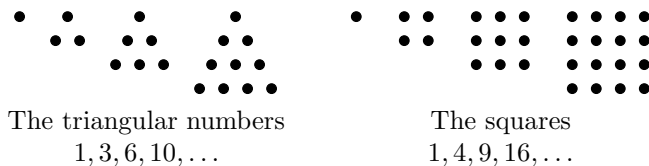
$n = 1$	1
$n = 2$	none; 12
$n = 3$	231; 132; none
$n = 4$	none; 3412; 1423, 2413; 1324, 2314

Note that if we now list the *number* of zigzag permutations of a given length ending in a given digit, we get a triangular table that starts out 1 (row 1), 0, 1 (row 2), 1, 1, 0 (row 3), 0, 1, 2, 2, \dots . Look familiar? That's right, it's precisely the first four rows of the zigzag triangle! Indeed, the pattern continues, and in general, the k th entry of row n of the zigzag triangle gives the number of zigzag permutations of $1, \dots, n$ that end in k .

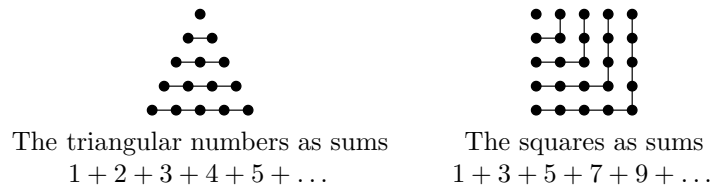
With that in mind, from the zigzag triangle, we see that there are 5 zigzags of length 5 that end in the number 1, 5 that end in 2, 4 that end in 3, 2 that end in 4, and 0 that end in 5. Can you find all 16 of the zigzag permutations of 1, 2, 3, 4, 5?⁶

6 Triangles, Squares, Pentagons, Hexagons, and Euler's Partition Formula

As Nerds, you have almost certainly thought about the *triangular numbers* and the *squares*:

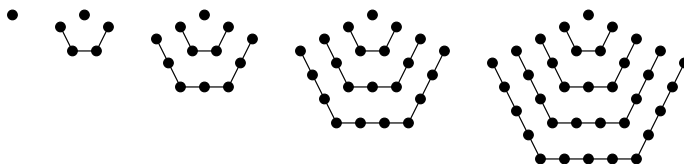


You may have also noticed that both the triangular numbers and the squares are sums of numbers in a particular arithmetic progression:



⁶They are, ordered by final digit: 24351, 25341, 34251, 35241, 45231; 14352, 15342, 34152, 35142, 45132; 14253, 15243, 24153, 25143; 13254, 23154.

More generally, for $p = 3, 4, 5, \dots$, we may define the p -gonal numbers to be the sums of the arithmetic progression that begins $1, p - 1, 2p - 3, \dots$. For example, the *pentagonal numbers* are the sequence $1, 1 + 4 = 5, 1 + 4 + 7 = 12$, and so on, and they can be drawn as follows:



It's actually not too hard to come up with a formula for the n th pentagonal number; just remember that the sum of a finite arithmetic progression is the number of terms times the average of the first and last term. Therefore, since the first term of $1, 4, 7, \dots$ is 1, and the k th term is $1 + 3(k - 1) = 3k - 2$, we see that the k th pentagonal number, which is the sum of the first k terms of $1, 4, 7, \dots$, is equal to

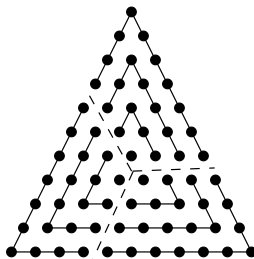
$$k \left(\frac{1 + (3k - 2)}{2} \right) = \frac{k(3k - 1)}{2}.$$

In fact, since this formula produces an integer for both positive and negative integer values of k , we may use this formula to *define* the (-1) st pentagonal number, the (-2) nd pentagonal number, the (-3) rd, and so on. For example, writing out the k th pentagonal number for $-4 \leq k \leq 4$, we get:

$$\dots \quad 26 \quad 15 \quad 7 \quad 2 \quad 0 \quad 1 \quad 5 \quad 12 \quad 22 \quad \dots$$

Can you see what the 5th and (-5) th pentagonal numbers are?⁷

Pentagonal numbers also have some more surprising properties. For example, comparing the following picture of triangle numbers and the preceding picture of the pentagonal numbers, we see that every pentagonal number is one-third of a triangular number:



Algebraically, we can see this by noting that 3 times the k th pentagonal number is

$$3 \left(\frac{k(3k - 1)}{2} \right) = \frac{(3k)(3k - 1)}{2},$$

⁷The 5th and (-5) th pentagonal numbers are 35 and 40, respectively.

which is the $(3k - 1)$ th triangle number, when $k > 0$. (Can you see why? What's a formula for the n th triangle number?) In fact, even if k is negative, $\frac{(3k)(3k - 1)}{2} = \frac{(3(-k))(3(-k) + 1)}{2}$ is the $(-k)$ th triangle number, and conversely, it's not much harder to see that if a triangle number is divisible by 3, then when you divide it by 3, you get a pentagonal number, for either positive or negative k .

Another, more profound, property of pentagonal numbers was discovered by the great 18th century mathematician Leonhard Euler. To explain this property, we need to describe the sequence of *partition numbers*. This is the sequence $p(1)$, $p(2)$, $p(3)$, etc., where $p(n)$ is the number of ways you can express the integer n as a sum of positive integers. For example, since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1, \text{ and}$$

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

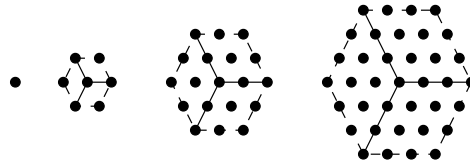
$p(4) = 5$ and $p(5) = 7$.

In these terms, Euler's *pentagonal number theorem* says that for $n > 1$,

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - \dots,$$

where the sum runs over all pentagonal numbers less than n , including those coming from $k < 0$. The signs alternate in the pattern $+$, $+$, $-$, $-$, and so on.

Incidentally, the hexagonal numbers 1, 6, 15, 28, ... defined above may not be the numbers that you would think of first when you think about hexagons; you may first think of the *hex numbers* 1, 7, 19, 37, ...:



It is an amusing fact that the sum of the first n hex numbers is precisely n^3 (try it!). Why? Well, every hex number hexagon can be seen as a corner of a cube, as indicated by the lines in the above hexagons. By nesting the first n of these "corners", one inside another, we can assemble a complete $n \times n \times n$ cube, which of course contains n^3 dots.

7 Moessner's Magic Sequence Generator; Sums of Powers and Bernoulli (or really, Faulhaber) numbers

Next, we consider a discovery made by mathematician Alfred Moessner in the 1950's. In the simplest case of *Moessner's magic*, start by taking the positive

integers and boxing the even ones:

$$1 \quad \boxed{2} \quad 3 \quad \boxed{4} \quad 5 \quad \boxed{6} \quad 7 \quad \boxed{8} \quad 9 \quad \boxed{10} \quad 11 \quad \boxed{12}$$

Now look at what happens when you sum up the unboxed terms, one by one. (For reasons of uniformity, we also box these final results.)

$$\begin{array}{cccccccccccc} 1 & \boxed{2} & 3 & \boxed{4} & 5 & \boxed{6} & 7 & \boxed{8} & 9 & \boxed{10} & 11 & \boxed{12} \\ \boxed{1} & & \boxed{4} & & \boxed{9} & & \boxed{16} & & \boxed{25} & & \boxed{36} & \end{array}$$

You get the squares, for exactly the same reasons ($1+3+5+\dots$) as the previous section.

Suppose that instead, we start by boxing every integer divisible by 3:

$$1 \quad 2 \quad \boxed{3} \quad 4 \quad 5 \quad \boxed{6} \quad 7 \quad 8 \quad \boxed{9} \quad 10 \quad 11 \quad \boxed{12}$$

Summing up terms, like before, we get:

$$\begin{array}{cccccccccccc} 1 & 2 & \boxed{3} & 4 & 5 & \boxed{6} & 7 & 8 & \boxed{9} & 10 & 11 & \boxed{12} \\ 1 & 3 & & 7 & 12 & & 19 & 27 & & 37 & 48 & \end{array}$$

If we now box every integer coming right before a box in the first row, and take sums again, we get:

$$\begin{array}{cccccccccccc} 1 & 2 & \boxed{3} & 4 & 5 & \boxed{6} & 7 & 8 & \boxed{9} & 10 & 11 & \boxed{12} \\ 1 & \boxed{3} & & 7 & \boxed{12} & & 19 & \boxed{27} & & 37 & \boxed{48} & \\ \boxed{1} & & & \boxed{8} & & & \boxed{27} & & & \boxed{64} & & \end{array}$$

That is, we get the cubes.

Can you guess what happens if we start by boxing every fourth integer and proceed as before? Try it!⁸

Indeed, Moessner's magic can be used to produce not only sequences of n th powers, but also many other similar sequences. For example, if we begin by boxing the triangular numbers $1, 3, 6, 10, \dots$ and take sums as before:

$$\begin{array}{cccccccccccc} \boxed{1} & 2 & \boxed{3} & 4 & 5 & \boxed{6} & 7 & 8 & 9 & \boxed{10} \\ & \boxed{2} & & 6 & \boxed{11} & & 18 & 26 & \boxed{35} & & \\ & & & \boxed{6} & & & 24 & \boxed{50} & & & \\ & & & & & & \boxed{24} & & & & \end{array}$$

We get the *factorial numbers* $1 = 1!, 2 = 2!, 6 = 3!, 24 = 4!, \dots$

⁸We get:

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & \boxed{4} & 5 & 6 & 7 & \boxed{8} & 9 & 10 & 11 & \boxed{12} \\ 1 & 3 & \boxed{6} & & 11 & 17 & \boxed{24} & & 33 & 43 & \boxed{54} & \\ 1 & \boxed{4} & & & 15 & \boxed{32} & & & 65 & \boxed{108} & & \\ \boxed{1} & & & & \boxed{16} & & & & \boxed{81} & & & \end{array}$$

In other words, we get the sequence of fourth powers.

In general, the pattern is:

If we start by boxing: $1 + 1, 2 + 2, 3 + 3, \dots$
the final result is: $1 \times 1, 2 \times 2, 3 \times 3, \dots$

If we start by boxing: $1 + 1 + 1, 2 + 2 + 2, 3 + 3 + 3, \dots$
the final result is: $1 \times 1 \times 1, 2 \times 2 \times 2, 3 \times 3 \times 3, \dots$

If we start by boxing: $1, 1 + 2, 1 + 2 + 3, \dots$
the final result is: $1, 1 \times 2, 1 \times 2 \times 3, \dots$

And so on.

Speaking of sums, you may remember that we've already discussed a formula for $1 + \dots + n$ (the triangular numbers):

$$1 + 2 + \dots + n = \frac{1}{2}(n^2 + n).$$

But what if we consider sums of squares or cubes? Well, we also get similar formulas:

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{3} \left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \right),$$

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4} (n^4 + 2n^3 + n^2).$$

The general pattern seems to be that there are formulas that look like:

$$1^{k-1} + 2^{k-1} + \dots + n^{k-1} = \frac{1}{k} \left(n^k + kn^{k-1} \cdot \frac{1}{2} + \dots \right),$$

but the question remains, what patterns can we find in the coefficients on the right-hand side? The answer was first studied by the 17th century mathematician Johann Faulhaber, and later given a more thorough treatment in the *Ars Conjectandi* of the 18th century mathematician Jacob Bernoulli.

Faulhaber's answer, as discussed by Bernoulli, says: Let

$$B^0 = 1, \quad B^1 = \frac{1}{2}, \quad B^2 = \frac{1}{6}, \quad B^4 = -\frac{1}{30},$$

$$B^6 = \frac{1}{42}, \quad B^8 = -\frac{1}{30}, \quad B^{10} = \frac{5}{66}, \quad \dots$$

and let $B^3 = B^5 = B^7 = \dots = 0$.⁹ Note that we write the terms in the sequence B^0, B^1, B^2, \dots as if they were powers, even though they aren't, for reasons that we hope will become clear in a moment.

⁹It is more standard to use the value $B^1 = -\frac{1}{2}$, but our choice of $+\frac{1}{2}$ is harmless, and make certain formulas more aesthetically pleasing to our eyes; see footnote 10.

In any case, if we adopt this strange manner of writing the sequence of *Bernoulli numbers*, then we get the following wonderful formula:

$$1^{k-1} + \dots + n^{k-1} = \frac{1}{k} \left((n+B)^k - B^k \right).$$

In other words,

$$\begin{aligned} 1^{k-1} + \dots + n^{k-1} &= \frac{1}{k} \left(n^k + \binom{k}{1} n^{k-1} B^1 + \binom{k}{2} n^{k-2} B^2 + \dots + \binom{k}{k-1} n^1 B^{k-1} \right) \\ &= \frac{1}{k} \left(n^k + k n^{k-1} \cdot \frac{1}{2} + \binom{k}{2} n^{k-2} \cdot \frac{1}{6} + \dots \right). \end{aligned}$$

Note that we expand “ $(n+B)^k - B^k$ ” using the Binomial Theorem and our strange way of writing terms from the sequence of Bernoulli numbers. For example, taking $k=4$, we get

$$\begin{aligned} (n+B)^4 - B^4 &= n^4 + 4n^3 B^1 + 6n^2 B^2 + 4n B^3 + B^4 - B^4 \\ &= n^4 + 4n^3 \cdot \frac{1}{2} + 6n^2 \cdot \frac{1}{6} + 4n \cdot 0 \\ &= n^4 + 2n^3 + n^2, \end{aligned}$$

which means that

$$1^3 + 2^3 + \dots + n^3 = \frac{1}{4}(n^4 + 2n^3 + n^2),$$

as we saw before.

The Bernoulli numbers have many other notable and important properties. For example, they can be computed using the startling formula “ $(B-1)^k = B^k$ ” for $k \geq 2$:

$$\begin{array}{ll} B^2 - 2B^1 + 1 = B^2 & \text{so we can solve for } B^1, \\ B^3 - 3B^2 + 3B^1 - 1 = B^3 & \text{so we can solve for } B^2, \\ B^4 - 4B^3 + 6B^2 - 4B^1 + 1 = B^4 & \text{so we can solve for } B^3, \end{array}$$

and so on.¹⁰ There is also the surprising fact, discovered by 19th century mathematicians Karl von Staudt and Thomas Clausen, that if $2, 3, \dots$ are all of the primes p with the property that $p-1$ divides $2n$, then

$$B^{2n} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p} + \dots$$

is an integer (in fact, it is equal to 1 for $2n \leq 12$).

Really, we could go on and write an entire separate article, or even a book, about the Bernoulli numbers — but we’ll end our own little sequence here.

¹⁰ If we use the more standard $B^1 = -\frac{1}{2}$, this formula becomes “ $(B+1)^k = B^k$ ”, but the idea is the same.

8 Some Places Where You Can Learn More (Did you notice the pattern in the titles of the sections of this article?)

This article is based on a lecture given by JHC at Santa Clara University in March 2005. If you want to find out more about the sequences we have discussed here, much of the material from this article and the lecture on which it based can be found in *The Book of Numbers*, by JHC and Richard K. Guy [2]. Interested Nerds should definitely continue their exploration of sequences there.

Another great resource of a different kind is the *On-Line Encyclopedia of Integer Sequences*, or OEIS, which can be found at:

www.research.att.com/~njas/sequences

The basic idea of this page is that if you enter the first few terms of a sequence, the OEIS will return all of the sequences it can think of that contain those terms at the beginning. (As a lower priority, the OEIS will also return sequences that contain those terms somewhere after the beginning.) Try “3, 1, 4, 1, 5, 9”, “2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1”, “6, 28, 496”, or “14, 18, 23, 28, 34, 42”. You can also search sequences by keyword; try “pi” or “divisor”. The OEIS is designed to be used by researchers in mathematics, so the entries tend to be quite condensed, to the point of sometimes being a bit cryptic; however, most Nerds should be able to get some enjoyment out of it. There are also a few places in the OEIS that give more extensive descriptions; for example,

www.research.att.com/~njas/sequences/classic.html

gives a very nice description of the Wythoff Array.

More advanced Nerds might also try looking at Ekhad and Zeilberger’s proof of the Cosmological Theorem [3] (code available from web page) and Calkin and Wilf’s description of the Stern sequence [1], both of which are available online (see references). Finally, a truly adventurous (or college-level) Nerd with access to both the Internet and a good college library might try looking at the homepage of Clark Kimberling and the references contained therein:

<http://faculty.evansville.edu/ck6/>

Kimberling has written extensively on the Wythoff Array and related phenomena, such as *interspersions* and *fractal sequences*. The scholarly college-level Nerd might do well to start with the articles on interspersions by Kimberling [6] and by Fraenkel and Kimberling [4].

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