A NON-RESIDUALLY FINITE SQUARE OF FINITE GROUPS

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ABSTRACT. We construct a non-positively curved non-residually finite square of finite groups whose vertex groups are of order 288, 288, 576, and 576. (In contrast, an earlier such example constructed by the authors had vertex groups of order between 2^{60} and 2^{150}.) In doing so, we demonstrate a new, more geometric method for embedding the fundamental group of a complete squared complex in the fundamental group of a square of finite groups.

1. INTRODUCTION

A *triangle* of groups is a diagram of group inclusions like the diagram T shown in Figure 1. The *fundamental group* of T, or $\pi_1(T)$, is the colimit of the diagram T. In other words, $\pi_1(T)$ is the group given by the presentation whose generators are the elements of X, Y, and Z, and whose relators are the multiplication tables for X, Y, and Z and the relations induced by the inclusions in the diagram. Note that in general, X, Y, and Z will not be subgroups of $\pi_1(T)$, since the other relations may make these groups collapse.

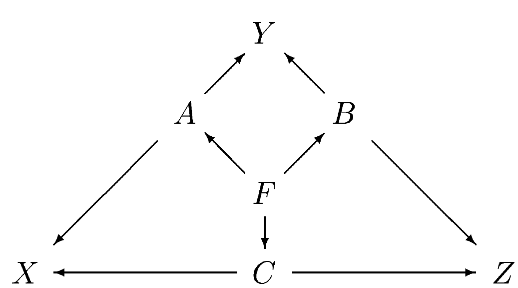

FIGURE 1. A triangle of groups T

Triangles of groups, especially triangles of finite groups, arise naturally in many places. For instance, many buildings can be formed from triangles of groups (see Brown [4], Ronan [15], and Tits [18]). In another context, triangles of groups play a key role in the proof of "uniqueness theorems" for sporadic finite simple groups (see Aschbacher and Segev [2] and Ivanov [10]).

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Geometrically, if T does not collapse (i.e., if X, Y, and Z are subgroups of $\pi_1(T)$), we see that $\pi_1(T)$ naturally acts on a triangulated complex, with quotient a triangle. This complex is called the *universal cover* of T, or $\widetilde{T}$. In that case, for any triangle Δ in the universal cover of T, the stabilizers of the vertices of Δ will be (isomorphic to) X, Y, and Z. Likewise, the stabilizers of the edges of Δ will be A, B, and C, and the stabilizer of all of Δ will be F.

One important situation where we are guaranteed to avoid collapse is the class of *non-positively curved triangles of groups*, as defined by Stallings [17]. More generally, there is a theory of non-positively curved *orbihedra* of groups, due to Haefliger [7], which deals with a diagram of inclusions corresponding to the cells of an arbitrary complex. In particular, one can consider non-positively curved squares, pentagons, etc., of groups. Note that for $n \geq 4$, it is very easy to construct lots of non-positively curved n-gons of groups; for instance, any n-gon of groups ($n \geq 4$) whose vertex groups are semidirect products of its edge groups is non-positively curved, since the Gersten-Stallings angle at each vertex is $\pi/2$. (See Stallings [17] for the definition of the Gersten-Stallings angle.)

Since the fundamental group of a (nondegenerate) non-positively curved polygon of finite groups is always infinite, it is natural to ask:

> *Which finiteness properties does the fundamental group of a non-positively curved polygon of finite groups have? For instance, is such a group residually finite, or virtually torsion-free?*

In particular, until recently, it was not known whether every non-positively curved polygon of finite groups has a residually finite fundamental group. Some positive results concerning the residual finiteness of certain classes of polygons of groups are due to Allenby and Tang [1] and Kim [11]. More recently, the authors [9] obtained non-positively curved squares and triangles of finite groups with non-residually finite fundamental groups.

Now, philosophically, the main idea behind [9] is that, just as the fundamental groups of finite 1-complexes are almost the same as amalgamated free products of finite groups, the fundamental groups of finite non-positively curved squared complexes are almost the same as those of non-positively curved squares of finite groups. In this paper, we illustrate this idea by showing, in a very natural way:

Theorem 1.1. *There exists a non-positively curved square of groups whose vertex groups have order 288, 288, 576, and 576, and whose fundamental group is non-residually finite.*

In contrast, our previous example of a non-positively curved non-residually square of finite groups [9] had vertex groups of order between 2^{60} and 2^{150}.

After introducing a certain complete squared complex X whose fundamental group has useful subgroup inseparability properties (Section 2), we describe a natural way of embedding $\pi_1(X)$ in a square of groups whose edge groups are free, and whose vertex groups are semidirect products (Section 3). An algebraic trick then allows us to make the edge and vertex groups finite (Section 4). Note that compared to our earlier embedding trick (the First Main Theorem in [9]), the embedding here is more natural, more geometric, and gives slightly nicer looking results in terms of the induced map on the universal cover.

In any case, our embedding gives us a non-positively curved square of (small) finite groups R with the same subgroup inseparability property that X has, so by "doubling" R, we obtain a nonresidually finite example (Section 5). In conclusion, we list some related open problems of interest (Section 6).

2. The complete squared complex X

In this section, we briefly describe some material that we will need from [19]. The interested reader can find a more elaborate discussion there.

Definition 2.1. A combinatorial 2-complex Y whose 2-cells are squares is said to be a *complete squared complex* (CSC) if the link of each vertex of Y is a complete bipartite graph.

The simplest example of a CSC is the direct product of two graphs. An equivalent definition to the one given above is that a squared 2-complex is a CSC if it is locally isomorphic to the direct product of two trees. In fact, it is easy to prove that:

Theorem 2.2. *Let Y be a squared 2-complex, and let $\widetilde{Y}$ denote the universal cover of Y. Then Y is a CSC if and only if $\widetilde{Y}$ is isomorphic to the direct product of two trees.*

We now give an example of a CSC X which is not the direct product of two graphs. X may be described as the result of gluing 6 squares to a bouquet of 5 edges in the manner defined by Figure 2. We think of the 1-skeleton of X as the union of a bouquet H of 2 horizontal circles, labelled $\{x, y\}$, and a bouquet V of 3 vertical circles, labelled $\{a, b, c\}$.

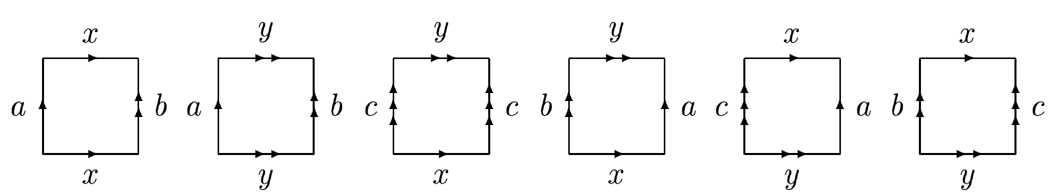

Figure 2. The six squares of X

Since X has only one 0-cell, $\pi_1(X)$ is generated by $\{a, b, c, x, y\}$, and we may read the following relators for $\pi_1(X)$ off of Figure 2.

$$\begin{array}{lccc} (2.1) & xb = ax & yb = ay & xc = cy \\ & xa = by & ya = cx & yc = bx \end{array}$$

Because of Theorem 2.2, $\widetilde{X}$ is isomorphic to a direct product of trees $\widetilde{V} \times \widetilde{H}$. Accordingly, we obtain the following corollary for use in Section 3.

Corollary 2.3. *Each element of $\pi_1(X)$ can be expressed as a product $v \cdot h$, where v and h are reduced combinatorial paths in V and H, respectively (i.e., $v \in \pi_1(V)$ and $h \in \pi_1(H)$).*

To describe the most useful property of the complex X (Theorem 2.5), we need the following definition.

Definition 2.4. A group G is said to be *subgroup separable* with respect to a subgroup C provided that for each element $g \in G - C$, there is a finite index subgroup $K \subset G$ containing C such that $g \notin K$. A group which is subgroup separable with respect to the trivial subgroup is said to be *residually finite*.

Equivalently, G is subgroup separable with respect to C if, for every $g \in G - C$, there is some finite quotient of G in which the image of g is not contained in the image of C.

Theorem 2.5. *The group $\pi_1(X)$ is not subgroup separable with respect to $\langle a, b, c \rangle$. Specifically, the element $xy^{-1} \notin \langle a, b, c \rangle$ cannot be separated from $\langle a, b, c \rangle$ in any finite quotient of $\pi_1(X)$.* □

The proof of Theorem 2.5 is based on the following contradiction. On the one hand, careful analysis of covers of the complex X shows that if $\langle a, b, c \rangle$ could be separated from xy^{-1} in a finite quotient, then X would have a finite cover which is the direct product of two graphs. On the other hand, one can deduce from certain aperiodic tilings called *anti-tori* which occur in $\widetilde{X}$, that X has *no* such finite cover which is a product. More specifically, there is a particular anti-torus in $\widetilde{X}$ which implies that that a and y do not have commuting non-trivial powers a^n, y^m. Since a and y are vertical and horizontal, this clearly precludes the existence of a finite product cover.

3. A "free" embedding of $\pi_1(X)$

We now wish to transfer the subgroup inseparability property of $\pi_1(X)$ to the fundamental group of some non-positively curved square of groups R by embedding $\pi_1(X)$ in $\pi_1(R)$. The idea behind this embedding may be thought of in terms of an equivariant map from $\widetilde{X}$ to $\widetilde{R}$. We send each 1-cell in $\widetilde{X}$ to a pair of opposite "wall-crossing paths" in $\widetilde{R}$. In other words, each generator of $\pi_1(X)$ is sent to the product of a pair of elements stabilizing opposing edges of R. This idea is depicted in Figure 3, where the heavy solid lines represent a typical square of $\widetilde{X}$, the light solid lines represent our subdivision of this square, and the dashed lines represent a portion of $\widetilde{R}$. (The annotations in Figure 3 will be explained below.)

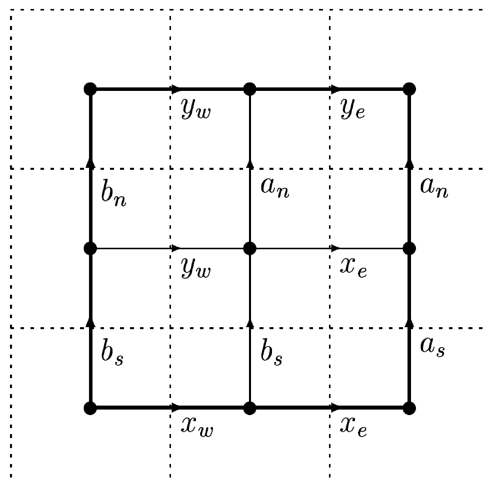

FIGURE 3. Embedding picture/van Kampen diagram for the fourth square of X

We can implement this idea algebraically in the following way. Let

$$E = \langle x_e, y_e \rangle \qquad N = \langle a_n, b_n, c_n \rangle$$
$$W = \langle x_w, y_w \rangle \qquad S = \langle a_s, b_s, c_s \rangle \tag{3.1}$$

be free groups on the indicated generating sets, and let R be the square of groups given by:

$$\begin{array}{ccccc} N \rtimes W & \leftarrow & N & \rightarrow & E \rtimes N \\ \uparrow & & \uparrow & & \uparrow \\ W & \leftarrow & 1 & \rightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ W \rtimes S & \leftarrow & S & \rightarrow & S \rtimes E \end{array} \tag{3.2}$$

Note that, as mentioned in the introduction, R is non-positively curved.

The semidirect products which form the vertex groups of R are defined by relations described by the following mnemonic, with one relation per vertex group per 2-cell of X.

- The relation of $S \rtimes E$ arising from each 2-cell of X imitates the way the west edge of the 2-cell is changed when it is pushed through the South edge to the East edge.

- The relation of $E \rtimes N$ arising from each 2-cell of X imitates the way the south edge of the 2-cell is changed when it is pushed through the East edge to the North edge.
- The relation of $N \rtimes W$ arising from each 2-cell of X imitates the way the West edge of the 2-cell is changed when it is pushed through the North edge to the east edge.
- The relation of $W \rtimes S$ arising from each 2-cell of X imitates the way the South edge of the 2-cell is changed when it is pushed through the West edge to the north edge.

More precisely, these relations are given by the following array, where the rows correspond with the six 2-cells of X, in the order given in Figure 2, and the columns correspond to the vertex groups $S \rtimes E$, $E \rtimes N$, $N \rtimes W$, and $W \rtimes S$, in that order.

$$\begin{array}{lllll}
(3.3) & x_e^{-1}a_s x_e = b_s & b_n^{-1}x_e b_n = x_e & x_w^{-1}a_n x_w = b_n & a_s^{-1}x_w a_s = x_w \\
(3.4) & y_e^{-1}a_s y_e = b_s & b_n^{-1}y_e b_n = y_e & y_w^{-1}a_n y_w = b_n & a_s^{-1}y_w a_s = y_w \\
(3.5) & x_e^{-1}c_s x_e = c_s & c_n^{-1}x_e c_n = y_e & y_w^{-1}c_n y_w = c_n & c_s^{-1}x_w c_s = y_w \\
(3.6) & x_e^{-1}b_s x_e = a_s & a_n^{-1}x_e a_n = y_e & y_w^{-1}b_n y_w = a_n & b_s^{-1}x_w b_s = y_w \\
(3.7) & y_e^{-1}c_s y_e = a_s & a_n^{-1}y_e a_n = x_e & x_w^{-1}c_n x_w = a_n & c_s^{-1}y_w c_s = x_w \\
(3.8) & y_e^{-1}b_s y_e = c_s & c_n^{-1}y_e c_n = x_e & x_w^{-1}b_n x_w = c_n & b_s^{-1}y_w b_s = x_w
\end{array}$$

Now let ϕ be the map from $\pi_1(X)$ to $\pi_1(R)$ defined by

$$\begin{array}{c} \phi(a) = a_s a_n \qquad \phi(b) = b_s b_n \qquad \phi(c) = c_s c_n \\ \phi(x) = x_w x_e \qquad \phi(y) = y_w y_e. \end{array} \tag{3.9}$$

We first claim that ϕ is a homomorphism. To verify this claim, it is enough to verify that the defining relations (2.1) for $\pi_1(X)$ are respected by ϕ. It is not too hard to do this directly, as each row of (3.3)–(3.8) yields the fact that ϕ respects the corresponding defining relation in (2.1). More instructively, consider Figure 3 again. The van Kampen diagram contained in Figure 3 shows that the relation from square number 4 is respected by ϕ, and similar van Kampen diagrams show that the other relations are respected. (Note that the southeast corner of the van Kampen diagram in Figure 3 is obtained from relation number 4 for $S \rtimes E$, the northeast corner from $E \rtimes N$, and so on.)

Perhaps more importantly, Figure 3 shows that ϕ induces a map from $\widetilde{X}$ to $\widetilde{R}$ which is an embedding on each square of $\widetilde{X}$, so at this point, one might optimistically hope that ϕ actually induces an embedding of all of $\widetilde{X}$ in $\widetilde{R}$. However, the induced map on universal covers is not an embedding, or even a local embedding, for the following reason.

Let Γ be the figure eight graph, and let Γ' be the edge of groups with trivial edge group and both vertex groups isomorphic to Z_2^2 (where Z_2 denotes the group of order 2). Let one vertex group of Γ' be denoted by $\langle a, b\rangle$, and let the other be $\langle c, d\rangle$. Recall that $\pi_1(\Gamma) = F_2 = \langle x, y\rangle$, the free group on two generators, and $\pi_1(\Gamma') = Z_2^2 * Z_2^2$. Consider the homomorphism $\rho : \pi_1(\Gamma) \to \pi_1(\Gamma')$ defined by $\rho(x) = ac$, $\rho(y) = bd$. Now, as shown in Figure 4, ρ naturally induces a map from $\widetilde{\Gamma}$ (dotted curves on the left side of Figure 4) to $\widetilde{\Gamma}'$ (solid lines on the right side of Figure 4). However, the map on universal covers is not an embedding, since it is not a local embedding in the neighborhood of a 0-cell of $\widetilde{\Gamma}$, as can be seen in Figure 4. The analogous local problem occurs with the induced map on universal covers $\widetilde{X} \to \widetilde{R}$, and so this induced map is also not an embedding.

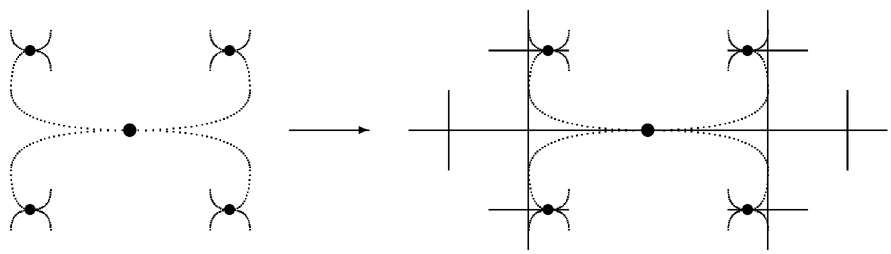

FIGURE 4. The induced map on universal covers

Nevertheless, we can see that ϕ itself is still an embedding as follows: First, consider the free products $N * S$ and $W * E$ which correspond to obvious suborbihedra of R. We claim that the intersection of the corresponding subgroups of $\pi_1(R)$ is trivial. To see this, observe that an element of $N * S$ must stabilize the associated vertical tree subspace of $\widetilde{R}$, and similarly, an element of $W * E$ must stabilize the associated horizontal tree subspace of $\widetilde{R}$. An element in both subgroups must stabilize both trees, and therefore must stabilize their intersection, which is the center of a square, since $\widetilde{R}$ is a product. Therefore, since the center of a square has trivial stabilizer, an element in both subgroups must be trivial.

Next, observe that ϕ maps $\pi_1(V)$ to $N * S$ and ϕ maps $\pi_1(H)$ to $W * E$. Furthermore, by the normal form theorem for free products, both of these (restricted) maps are injections, just as in the case of $\pi_1(\Gamma) \to \pi_1(\Gamma')$ above.

So now suppose that $\phi(g) = 1$ for some $g \in \pi_1(X)$. By Corollary 2.3, $g = vh^{-1}$ for some elements $v \in \pi_1(V)$ and $h \in \pi_1(H)$, which means that $\phi(v) = \phi(h)$. However, since $\phi(v) \in N * S$ and $\phi(h) \in W * E$, from the above argument, $\phi(v) = 1 = \phi(h)$. Therefore, since ϕ is injective on $\pi_1(V)$ and $\pi_1(H)$, we conclude that $v = 1$ and $h = 1$, and so $g = vh^{-1} = 1$.

We therefore obtain:

Theorem 3.1. *ϕ embeds $\pi_1(X)$ as a subgroup of $\pi_1(R)$.* □

Remark 3.2. Although we did not obtain an equivariant embedding of $\widetilde{X}$ in $\widetilde{R}$, we note that because the vertex groups are semidirect products of the edge groups, it follows that $\widetilde{R}$ is the product of two trees. Again, this shows how the geometry of $\widetilde{X}$ is duplicated in $\widetilde{R}$. Furthermore, we note that ϕ actually induces a quasi-isometry.

Finally, since Theorem 2.5 implies that xy^{-1} cannot be separated from $\langle a, b, c\rangle$ in any finite quotient of $\pi_1(X)$, $\phi(xy^{-1})$ cannot be separated from $\langle \phi(a), \phi(b), \phi(c)\rangle$ in any finite quotient of $\pi_1(R)$. Let C denote the amalgamated free product $(E \rtimes N) *_E (S \rtimes E)$ corresponding to the east edge of R. Since $\phi(xy^{-1}) \notin C$ and $\langle \phi(a), \phi(b), \phi(c)\rangle \subseteq C$, we have:

Corollary 3.3. *Let C be the amalgamated free product corresponding to the east edge of R. Then $\pi_1(R)$ is not C-separable.* □

Remark 3.4. The construction in this section may be generalized in the following way. Note that (3.3)–(3.8) give complete definitions for $S \rtimes E$, $E \rtimes N$, $N \rtimes W$, and $W \rtimes S$ precisely because X is a CSC with just one 0-cell, which means that any "north" edge of X meets any "east" edge of X in precisely 1 square, and so on. In the general case of a CSC with more than one 0-cell, if we perform the same procedure, with the addition of commuting relations corresponding to edges which do not meet in a square, we get an injective homomorphism from the fundamental groupoid $\pi(X)$ into the fundamental group of a square of groups R, with the geometry shown above.

Remark 3.5. J. McCammond (personal communication) has observed that the group with defining relations (3.3)–(3.8) is a LOG group (as defined by Howie [8]). More generally, Remark 3.4 implies that the fundamental groupoid of any CSC may be embedded in a LOG group.

4. Making the embedding finite

Having embedded $\pi_1(X)$ in the fundamental group of a non-positively curved square of groups R, we now use an algebraic trick to construct a quotient $\overline{R}$ of R which is a non-positively curved square of *finite* groups.

The idea is that the semidirect products described by (3.3)–(3.8) really just describe how the action of E permutes the generators of S, the action of N permutes the generators of E, and so on. If we write these permutations down, for E and W, we have, in cycle form:

(4.1) $\quad x_e = (a_s, b_s) \qquad y_e = (a_s, b_s, c_s)$

(4.2) $\quad x_w = (a_n, b_n, c_n) \qquad y_w = (a_n, b_n),$

and for N and S, we have:

(4.3) $\quad a_n = (x_e, y_e) \qquad b_n = () \qquad c_n = (x_e, y_e)$

(4.4) $\quad a_s = () \qquad b_s = (x_w, y_w) \qquad c_s = (x_w, y_w).$

This is our first approximation for the edge groups of $\overline{R}$.

The problem with our first approximation is that the permutations indicated by x_e and y_e are no longer automorphisms of the group $\langle a_s, b_s, c_s\rangle$, and so on. However, we may fix this problem by "symmetrizing" the generators of the edge groups with respect to all possible permutations of the generators. That is, let

$$\begin{aligned} x_e &= ((a_s, b_s) \quad\ , (a_s, b_s, c_s)) \\ y_e &= ((a_s, b_s, c_s), (a_s, b_s) \quad\ ) \end{aligned} \tag{4.5}$$

be the indicated elements of $S_3 \times S_3$, where S_n denotes the symmetric group of degree n, and similarly, let

$$\begin{aligned} x_w &= ((a_n, b_n, c_n), (a_n, b_n) \quad\ ) \\ y_w &= ((a_n, b_n) \quad\ , (a_n, b_n, c_n)), \end{aligned} \tag{4.6}$$

$$\begin{aligned} a_n &= ((x_e, y_e), (x_e, y_e), () \quad\ , () \quad\ , (x_e, y_e), (x_e, y_e)) \\ b_n &= (() \quad\ , (x_e, y_e), (x_e, y_e), (x_e, y_e), (x_e, y_e), () \quad\ ) \\ c_n &= ((x_e, y_e), () \quad\ , (x_e, y_e), (x_e, y_e), () \quad\ , (x_e, y_e)), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} a_s &= (() \quad\ , () \quad\ , (x_w, y_w), (x_w, y_w), (x_w, y_w), (x_w, y_w)) \\ b_s &= ((x_w, y_w), (x_w, y_w), () \quad\ , (x_w, y_w), () \quad\ , (x_w, y_w)) \\ c_s &= ((x_w, y_w), (x_w, y_w), (x_w, y_w), () \quad\ , (x_w, y_w), () \quad\ ) \end{aligned} \tag{4.8}$$

be the indicated elements of either S_3^2 (in (4.6)) or S_2^6 (in (4.7) and (4.8)).

Let E, N, W, and S again be defined by (3.1), replacing the free generators with the permutations in (4.5)–(4.8), and let the semidirect products $S \rtimes E$, $E \rtimes N$, $N \rtimes W$, and $W \rtimes S$ again be given by (3.3)–(3.8). (Note that E acts on S, N acts on E, and so on, by factoring through the "erase all coordinates but the first" homomorphism.) A brief calculation shows that $E \cong W \cong S_3^2$, and $N \cong S \cong Z_2^3 \subseteq S_2^6$.

Finally, let $\overline{R}$ be the square of finite groups shown in (3.2). Repeating the arguments of the previous section, we see that (3.9) defines an injective homomorphism ϕ from $\pi_1(X)$ to $\pi_1(\overline{R})$. Therefore, summarizing the results of Sections 3 and 4, we have:

Theorem 4.1. *Let $\overline{R}$ be the non-positively curved square of finite groups whose edge and vertex groups are described by* (4.5)–(4.8) *and* (3.2)–(3.8), *and let C be the amalgamated free product corresponding to the east edge of $\overline{R}$. Then $\pi_1(\overline{R})$ is not C-separable.* □

Remark 4.2. Continuing the observation in Remark 3.4, we note that the "symmetrizing" idea in this section also works if we start with an arbitrary CSC. In particular, if the initial CSC has h horizontal and v vertical 1-cells, then E and W will be subgroups of $S_v^{h!}$, N and S will be subgroups of $S_h^{v!}$, and the vertex groups will be subgroups of either $S_v \wr S_h$ or $S_h \wr S_v$. These groups will generally be very large as h and v increase; however, since $h = 2$ and $v = 3$ for the complex X, our edge and vertex groups are relatively small.

5. Getting a non-residually finite example

To get our non-residually finite example, let $G = \pi_1(\overline{R})$, and let C be the amalgamated free product corresponding to the east edge of $\overline{R}$. Then, since G is not C-separable, it follows from Long and Niblo [13] (see also [19]) that $G *_C G$ (with both embeddings of C in G the same), the *double* of G along its subgroup C, is not residually finite. It therefore remains only to embed the double of G along C in an appropriate non-positively curved square of finite groups.

Now in general, the double $(A *_B C) *_C (A *_B C)$ of the group $A *_B C$ along C is an index 2 subgroup of $A *_B (C \times Z_2)$. In our case, G splits as $A *_B C$ where $A = (N \rtimes W) *_W (W \rtimes S)$, $B = N * S$, and $C = (E \rtimes N) *_E (S \rtimes E)$. Therefore, the double $G *_C G$ is an index 2 subgroup of the group $A *_B (C \times Z_2)$ which is the fundamental group of the non-positively curved square of groups D indicated in the diagram below.

$$
\begin{array}{ccccc}
N \rtimes W & \leftarrow & N & \rightarrow & (E \rtimes N) \times Z_2 \\
\uparrow & & \uparrow & & \uparrow \\
W & \leftarrow & 1 & \rightarrow & E \quad \times Z_2 \\
\downarrow & & \downarrow & & \downarrow \\
W \rtimes S & \leftarrow & S & \rightarrow & (S \rtimes E) \times Z_2
\end{array}
\tag{5.1}
$$

Since the orders of the vertex groups of D are $288 = 8{\cdot}36$, 288, $576 = 288{\cdot}2$, and 576, Theorem 1.1 follows.

6. Some related open problems

In closing, we mention several open problems about polygons of finite groups.

- *Are the fundamental groups of negatively curved polygons of finite groups residually finite?* Note that this is a special case of the question: *Are word hyperbolic groups residually finite?* (See [6].)
- *Are the fundamental groups of non-positively curved polygons of groups Hopfian?* Note that Meier [14] has proved this in the building sub-case, and Sela [16] proved this for word-hyperbolic groups. An example of a compact non-positively curved 2-complex with non-Hopfian fundamental group is given in [21].
- *Are the fundamental groups of non-positively curved polygons of finite groups virtually torsion-free?* That is, do such groups contain torsion-free subgroups of finite index? Note that examples of non-positively

curved orbihedra whose fundamental groups are not virtually torsion-free were given in [19], [20].

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