The Spanning-Linear Independence Duality

Given vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in \mathbb{R}^n \) and suppose \( \mathbf{A} = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k] \). We have two “dual” sets of equivalent statements:

a. \( \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \) spans \( \mathbb{R}^n \).

b. Each \( \mathbf{b} \in \mathbb{R}^n \) is a linear combination of the columns of \( \mathbf{A} \).

c. For every \( \mathbf{b} \in \mathbb{R}^n \), the matrix equation \( \mathbf{A}\mathbf{x} = \mathbf{b} \) has a solution.

d. For every \( \mathbf{b} \in \mathbb{R}^n \), the system with augmented matrix \( [\mathbf{A} \mathbf{b}] = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k \mathbf{b}] \) has a solution.

e. \( \mathbf{A} \) has a pivot position in every row.

f. The linear function \( T: \mathbb{R}^k \to \mathbb{R}^n \) defined by \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) is onto.

a. \( \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \) is linearly independent.

b. No \( \mathbf{a}_i \) is a linear combination of the remaining \( \mathbf{a}_j \)s.

c. The matrix equation \( \mathbf{A}\mathbf{x} = \mathbf{0} \) has only the trivial solution.

d. The system with augmented matrix \( [\mathbf{A} \mathbf{0}] = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_k \mathbf{0}] \) has only the trivial solution.

e. \( \mathbf{A} \) has a pivot position in every column.

f. The linear function \( T: \mathbb{R}^k \to \mathbb{R}^n \) defined by \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) is one-to-one.

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