Finding Linear Dependency Relationships Among Vectors in $\mathbb{R}^n$

As we have seen, the system of linear equations

$$
\begin{align*}
    a_{11}x_1 &+ a_{12}x_2 + \ldots + a_{1m}x_m = b_1 \\
    a_{21}x_1 &+ a_{22}x_2 + \ldots + a_{2m}x_m = b_2 \\
    \vdots & \quad \quad \quad \quad \quad \quad \quad \vdots \\
    a_{n1}x_1 &+ a_{n2}x_2 + \ldots + a_{nm}x_m = b_n \\
\end{align*}
$$

(1)

can be written as a single matrix-vector equation

$$
Ax = b,
$$

(2)

where

$$
A = \begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1m} \\
    a_{21} & a_{22} & \ldots & a_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \ldots & a_{nm}
\end{bmatrix}
$$

and

$$
b = \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_n
\end{bmatrix}.
$$

We solved (1) by row reducing the augmented matrix $[A \ b] = [a_1 \ldots a_m \ b]$, where $a_1, \ldots, a_m$ are the columns of $A$, as usual. In the process, we saw that if $[\tilde{A} \ \tilde{b}]$ is any row-reduced version of $[A \ b]$, then a solution $x$ of (2) is also a solution of the system whose augmented matrix is $[\tilde{A} \ \tilde{b}]$, i.e., the same $x$ will also solve

$$
\tilde{A}x = \tilde{b},
$$

(3)

Conversely, since all elementary row operations are reversible, any solution of (3) will also be a solution of (2).

If, in particular, $b_1 = 0, b_2 = 0 \ldots b_m = 0$, i.e., $b = \mathbf{0}$, we have a homogeneous system. In that case, $\tilde{b} = \mathbf{0}$ for $[\tilde{A} \ \mathbf{0}]$ any row-reduced version of $[A \ \mathbf{0}]$.

We also saw that (2) can be expressed as the vector equation

$$
x_1a_1 + x_2a_2 + \ldots + x_m a_m = b,
$$

(4)

where $a_i$ is the $i$-th column of $A$.

Combining the foregoing, we get that any solution $x$ of

$$
x_1a_1 + x_2a_2 + \ldots + x_m a_m = 0
$$

(5)

is a solution of

$$
x_1\tilde{a}_1 + x_2\tilde{a}_2 + \ldots + x_m \tilde{a}_m = 0
$$

(6)

and vice versa. This allows us to determine easily linear dependency relations among vectors in $\mathbb{R}^n$ as follows:
Suppose \( \{a_1, \ldots, a_m\} \subseteq \mathbb{R}^n \). We want to know whether or not this set is linearly dependent and, if so, what linear dependency relations hold among its elements.

1. Form the matrix \( A = [a_1, \ldots, a_m] \).

2. Reduce \( A \) to reduced row-echelon form \( \tilde{A} \) by elementary row operations.

3. The definition of reduced row-echelon form makes it easy to see how the non-pivot columns of \( \tilde{A} \) can be written as linear combinations of the pivot columns; we also see immediately that the pivot columns must be linearly independent. (See example below.)

4. By the discussion above, the same linear dependency relationships that hold—or fail to hold—among the columns of \( \tilde{A} \) must also hold—or fail to hold—among the corresponding columns of \( A \).

**Example 1.** Determine whether
\[
\begin{bmatrix}
2 \\ 1 \\ -3 \\ -1 \\ 2
\end{bmatrix},
\begin{bmatrix}
3 \\ 2 \\ -3 \\ 0 \\ 1
\end{bmatrix},
\begin{bmatrix}
0 \\ 1 \\ 3 \\ 3 \\ -4
\end{bmatrix},
\begin{bmatrix}
0 \\ 0 \\ 3 \\ 1 \\ -2
\end{bmatrix},
\begin{bmatrix}
9 \\ 4 \\ -3 \\ -2 \\ 3
\end{bmatrix}
\]
is linearly dependent

and, if so, find expressions of some of its elements as linear combinations of the others.

1. \( A = \begin{bmatrix} 2 & 3 & 0 & 0 & 9 \\ 1 & 2 & 1 & 0 & 4 \\ -3 & -3 & 3 & 3 & -3 \\ -1 & 0 & 3 & 1 & -2 \\ 2 & 1 & -4 & -2 & 3 \end{bmatrix} \)

2. The reduced row-echelon form of \( A \) is \( \tilde{A} = \begin{bmatrix} 1 & 0 & -3 & 0 & 6 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \)

3. We note that \( \tilde{a}_3 = -3\tilde{a}_1 + 2\tilde{a}_2 \) and \( \tilde{a}_5 = 6\tilde{a}_1 - \tilde{a}_2 + 4\tilde{a}_4 \) and that there is no linear dependency relationship among \( \tilde{a}_1, \tilde{a}_2 \) and \( \tilde{a}_4 \). The first two equations imply that \( 3\tilde{a}_1 - 2\tilde{a}_2 + \tilde{a}_3 = 0 \) and \( -6\tilde{a}_1 + \tilde{a}_2 - 4\tilde{a}_4 + \tilde{a}_5 = 0 \), which, in turn, imply that \( 3a_1 - 2a_2 + a_3 = 0 \) and \( -6a_1 + a_2 - 4a_4 + a_5 = 0 \).

4. So we conclude that \( a_3 = -3a_1 + 2a_2 \) and \( a_5 = 6a_1 - a_2 + 4a_4 \) and that there is no linear dependency relationship among \( a_1, a_2 \) and \( a_4 \) (otherwise there would have been such a relationship among \( \tilde{a}_1, \tilde{a}_2 \) and \( \tilde{a}_4 \)).