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Codimension one Anosov flows and a conjecture of Verjovsky

SLOBODAN SIMIĆ†
Department of Mathematics (M/C 249), University of Illinois at Chicago,
851 South Morgan Street, Chicago, IL 60607-7045, USA

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Abstract. Let Φ be a C^2 codimension one Anosov flow on a compact Riemannian manifold M of dimension greater than three. Verjovsky conjectured that Φ admits a global cross-section and we affirm this conjecture when Φ is volume preserving in the following two cases: (1) if the sum of the strong stable and strong unstable bundle of Φ is θ-Hölder continuous for all $\theta < 1$; (2) if the center stable bundle of Φ is of class $C^{1+\theta}$ for all $\theta < 1$. We also show how certain transitive Anosov flows (those whose center stable bundle is C^1 and transversely orientable) can be 'synchronized', that is, reparametrized so that the strong unstable determinant of the time t map (for all t) of the synchronized flow is identically equal to e^t. Several applications of this method are given, including vanishing of the Godbillon–Vey class of the center stable foliation of a codimension one Anosov flow (when $\dim M > 3$ and that foliation is $C^{1+\theta}$ for all $\theta < 1$), and a positive answer to a higher-dimensional analog to Problem 10.4 posed by Hurder and Katok in [**HK**].

1. *Introduction*

In the 1970s Alberto Verjovsky posed the following conjecture (also stated in [**Gh3**]).

Conjecture of Verjovsky. Every codimension one Anosov flow on a compact manifold of dimension greater than three admits a global cross-section.

By a celebrated result of Newhouse [**Nh**] and Franks [**Fra**], this conjecture implies that every codimension one Anosov flow on a compact manifold of dimension greater than three is topologically conjugate to the suspension of a linear toral automorphism. We will show that this is the case if certain regularity assumptions are imposed on the invariant Anosov bundles.

Recall that a nonsingular C^1 flow $\{f_t\}$ on a compact Riemannian manifold M is called *Anosov* if the tangent bundle of M splits continuously into three invariant

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† Present address: Department of Mathematics, University of Southern California, 1042 W. 36th Place, Los Angeles, CA 90089-1113, USA.

subbundles, $TM = E^c \oplus E^{ss} \oplus E^{uu}$, where E^c is the line bundle tangent to the flow, E^{ss} is exponentially contracted and E^{uu} is exponentially expanded by the flow in the forward direction. We call E^{ss} and E^{uu} the *strong stable* and *strong unstable bundle* (or distribution) respectively. We also have $E^{cs} = E^{ss} \oplus E^c$ and $E^{cu} = E^{uu} \oplus E^c$, called the *center stable* and *center unstable bundle* respectively. In general, the strong distributions are only θ-Hölder continuous for some $0 < \theta < 1$; however, all Anosov distributions are uniquely integrable giving rise to continuous foliations (denoted accordingly by W^{ss}, W^{uu}, W^{cs} and W^{cu}). For details on these basic results the reader should consult [**An**], [**Pl1**] and [**Sm**].

An Anosov flow is *of codimension one* if $\dim E^{ss} = 1$ or $\dim E^{uu} = 1$. When speaking of a codimension one Anosov flow, we will always assume that E^{uu} is one-dimensional.

Recall that a compact codimension one submanifold Σ of a manifold M is said to be a *(global) cross-section* for a flow $\{f_t\}$ on M if Σ intersects every orbit of the flow transversely. In that case, $\{f_t\}$ can be obtained by 'suspending' the Poincaré (or first return) map $\Sigma \to \Sigma$. For discussion on the existence of cross-sections we refer to [**Ch**], [**Fri**] and [**Sc**].

Plante proved the conjecture of Verjovsky for manifolds with solvable fundamental group (see [**Pl2**], [**Pl3**] and also [**Ar**]). Ghys [**Gh3**] showed that the conjecture of Verjovsky holds if $E^{su} = E^{ss} \oplus E^{uu}$ is of class C^1, which we generalized in [**Si1**] to the case when E^{su} is only Lipschitz.

This paper is organized as follows. In §2 we prove an auxiliary result that for all negative time, the E^{uu}-determinant of an Anosov flow (i.e. its expansion cocycle) varies along the strong unstable leaves by a bounded amount independent of time (Corollary 2.2 and Theorem 2.3). This is used in §3 to prove that the conjecture of Verjovsky holds if the flow is volume preserving and the bundle E^{su} is Lip– (Theorem 3.2). Recall that Lip– means θ-Hölder continuous for all $\theta < 1$. In §4 we show that it is possible to 'synchronize' the strong unstable determinant of certain types of Anosov flows (those whose center stable bundle is C^1 and transversely orientable), that is, make it identically equal to e^t for all t by suitably reparametrizing the flow (Theorem 4.4). Applications of synchronization are given in §5 and 6 including a generalization of a conjecture of Ghys from [**Gh3**]; namely, the conjecture of Verjovsky holds if the flow is volume preserving and E^{cs} is of class $C^{1+\text{Lip}-}$ (Theorem 6.2). We also discuss vanishing of the Godbillon–Vey class of W^{cs} for codimension one Anosov flows in higher dimensions. Corollary 6.4 gives a positive answer to a higher-dimensional analog of Problem 10.4 from [**HK**].

2. *Expansion cocycle*

Let $\{f_t\}$ be a codimension one Anosov flow on a compact manifold M. Without loss of generality we assume that all the Anosov distributions are orientable (otherwise, we pass to an appropriate finite cover of M). Choose a continuous Riemann structure $\langle \cdot, \cdot \rangle$ on M such that:

- $TM = E^{uu} \oplus E^{cs}$ is an orthogonal splitting;
- X (the Anosov vector field) has length 1.

Let Z be a nonvanishing C^∞ vector field on M which is everywhere transverse to E^{cs}. Let Y be the projection of Z on E^{uu} relative to the chosen Riemannian metric.

Then Y is a continuous, completely integrable nonvanishing vector field. If necessary, modify the metric in a continuous fashion to make Y a unit vector field and define a positive function $\lambda : M \times \mathbb{R} \to \mathbb{R}$ by

$$T_x f_t(Y_x) = \lambda(x, t) Y_{f_t x},$$

with $x \in M$ and $t \in \mathbb{R}$. We call λ the *expansion cocycle* of f_t (relative to the chosen Riemannian metric). Clearly, λ is continuous in x (we will show more later), C^1 in t, $\lim_{t\to\infty} \lambda(x, t) = \infty$, and $\lim_{t\to-\infty} \lambda(x, t) = 0$ uniformly in $x \in M$. In fact, there exist numbers $K > 0$ and $c > 0$ such that for all $x \in M$ and $t \geq 0$,

$$\lambda(x, t) \geq K e^{ct}.$$

Define a 1-form ω by

$$\mathrm{Ker}(\omega) = E^{cs}, \quad \omega(Z) = 1.$$

Since E^{cs} is of class C^1 (see [**Hb**]), so is ω. (That is the only reason we needed Z: to ensure that ω is smooth in a direction transverse to E^{cs}.) Note also that $\omega(Y) = 1$. (This is because $Z = Y + V$, where $V \in E^{cs}$ and $\omega(V) = 0$.) By the Frobenius theorem, ω divides $d\omega$, i.e. there exists a continuous 1-form η such that $d\omega = \eta \wedge \omega$. Define a function $u : M \to \mathbb{R}$ by

$$u(x) = \eta(X_x).$$

Then we have the following theorem.

THEOREM 2.1.
(a) *The function u is of class C^1.*
(b) *For all $x \in M$ and $t \in \mathbb{R}$ we have*

$$u(f_t x) = \left.\frac{d}{ds}\right|_t \log \lambda(x, s).$$

Therefore

$$\lambda(x, t) = \exp\left\{\int_0^t u(f_s x)\, ds\right\}. \tag{1}$$

(c) *Let c and K be as above. Then for every point $x \in M$ and every $\tau > 0$,*

$$\frac{1}{\tau}\int_0^\tau u(f_t x)\, dt \geq \frac{1}{\tau}\log K + c.$$

Proof. (a) We have

$$\begin{aligned} u &= \eta(X) \\ &= \eta(X)\omega(Z) - \eta(Z)\omega(X) \\ &= (\eta \wedge \omega)(X, Z) \\ &= d\omega(X, Z) \\ &= X(\omega(Z)) - Z(\omega(X)) - \omega([X, Z]) \\ &= -\omega([X, Z]) \end{aligned}$$

because $\omega(X) = 0$ and $\omega(Z) = 1$. Since both $[X, Z]$ and ω are of class C^1, part (a) follows.

(b) Note that $\omega(Tf_t(Z_x)) = \lambda(x, t)$. Using the properties of the Lie derivative (see, for instance, [**Wa**]), we obtain

$$\begin{aligned} u(x) &= d\omega(X, Z) \\ &= i_Z(i_X d + d i_X)\omega \\ &= i_Z(L_X\omega) \\ &= (L_X\omega)(Z) \\ &= \frac{d}{dt}\bigg|_0 [\omega(Tf_t(Z_x))] \\ &= \frac{d}{dt}\bigg|_0 \lambda(x, t). \end{aligned}$$

Clearly, $\lambda(x, t)$ is a multiplicative 1-cocycle over the Anosov flow, that is

$$\lambda(x, t + s) = \lambda(x, t)\lambda(f_t(x), s). \tag{2}$$

Differentiating the last equality with respect to s at zero, we obtain

$$u(f_t x) = \frac{d}{ds}\bigg|_t \log \lambda(x, s).$$

This clearly implies formula (1).

(c) Let x be an arbitrary point and $\tau > 0$. Then by (b), $\int_0^\tau u(f_t x)\,dt = \log \lambda(x, t) \geq \log K + \tau c$, as desired. We remark that this could be used to construct an everywhere positive function u' with properties analogous to those of u, but we postpone this until §4. ☐

COROLLARY 2.2. *There exists a constant* $\ell > 0$ *such that if* $y \in W^{uu}(x)$, *then for all* $t > 0$,

$$\exp\{-\ell d_u(x, y)\} \leq \frac{\lambda(x, -t)}{\lambda(y, -t)} \leq \exp\{\ell d_u(x, y)\}.$$

Here $d_u(x, y)$ denotes the distance between x and y along $W^{uu}(x)$.

Proof. Suppose $y \in W^{uu}(x)$. Then $d(f_{-s}x, f_{-s}y) \leq Ae^{-\mu s}d(x, y)$ for some $A, \mu > 0$ and all $s > 0$. Let $B = \sup_M \|du\|$. If $t > 0$, then (1) implies

$$\begin{aligned} \frac{\lambda(x, -t)}{\lambda(y, -t)} &= \exp\left\{\int_0^{-t} [u(f_s x) - u(f_s y)]\,ds\right\} \\ &\leq \exp\left\{\int_0^{t} |u(f_{-s} x) - u(f_{-s} y)|\,ds\right\} \\ &\leq \exp\left\{B\int_0^{t} d(f_{-s} x, f_{-s} y)\,ds\right\} \\ &\leq \exp\left\{AB\int_0^{t} e^{-\mu s} d_u(x, y)\,ds\right\} \\ &= \exp\left\{-\frac{ABd_u(x, y)}{\mu}(e^{-\mu t} - 1)\right\} \qquad (3) \\ &\leq e^{\ell d_u(x, y)}, \end{aligned}$$

where $\ell = AB/\mu$. Observe that (3) is increasing with respect to t. The $\geq$ part of the inequality follows by switching the roles of x and y. □

It is easy to see that our reasoning used in the proofs of Corollary 2.2 and Theorem 2.1 also proves the following result.

THEOREM 2.3. *Let* $\{f_t\}$ *be an Anosov flow whose center stable distribution is of class* C^θ *for some* $\theta \leq 1$. *Let* $\lambda(x,t)$ *be the determinant of* $T_x f_t$ *on* E^{uu}. *Then there exists an* $\ell > 0$ *such that for all* $x, y \in M$ *with* $y \in W^{uu}(x)$, *and all* $t > 0$,

$$\exp\{-\ell d_u(x,y)^\theta\} \leq \frac{\lambda(x,-t)}{\lambda(y,-t)} \leq \exp\{\ell d_u(x,y)^\theta\}.$$

Therefore, there is some uniformity in the behavior of $x \mapsto \lambda(x,t)$ along the strong unstable manifolds of the flow. It is natural to ask the following question: can the constant ℓ be made as close to 0 as we want by suitably reparametrizing the flow? Or even better: can the given flow be reparametrized to make the expansion cocycle of the new flow *independent* of x?

The answer is yes, as we will see in §4.

3. *Anosov flows for which* E^{su} *is Lip–*

The following simple lemma due to Ghys [**Gh3**] is crucial in the proof of the main theorem of this section, Theorem 3.2. It basically says that if R is a rectangle in $T_x M$ where M is a compact n-manifold admitting a volume preserving codimension one Anosov flow $\{f_t\}$ such that two sides of R are parallel to the strong stable space E^{ss}_x and the other two are parallel to the strong unstable space E^{uu}_x of f_t, then the area of $f_{-t}(R)$ tends to zero as $t \to \infty$, provided that $n > 3$.

LEMMA 3.1. *Let* $f : E_1 \to E_2$ *be a linear isomorphism of Euclidean spaces preserving the orthogonal splitting* $E_i = S_i \oplus U_i$, *where* $\dim U_i = 1$ $(i = 1, 2)$. *If* $\dim E_1 = n - 1$ *and* $\|f|_{S_1}\| = \nu$, *then for all* $w_s \in S_2$ *and* $w_u \in U_2$,

$$(\det f)\|f^{-1}(w_s \wedge w_u)\| \leq \nu^{n-3}\|w_s \wedge w_u\|.$$

Proof. Let $v_s \in S_1$, $v_u \in U_1$ be arbitrary. Choose unit vectors $e_3, \ldots, e_{n-1}$ in S_1 such that $e_1 = v_u, e_2 = v_s, e_3, \ldots, e_{n-1}$ is an orthogonal basis of E_1. Then we have

$$\begin{aligned} \|v_u \wedge v_s\| \cdot \det f &= \|f(e_1 \wedge \cdots \wedge e_{n-1})\| \\ &= \|f(v_u \wedge v_s)\| \prod_{i=3}^{n-1} \|f(e_i)\| \\ &\leq \nu^{n-3}\|f(v_u \wedge v_s)\|. \end{aligned}$$

To complete the proof, take $w_u = f(v_u)$, $w_s = f(v_s)$. □

Now we can prove the main result of this section.

THEOREM 3.2. *Let* $\{f_t\}$ *be a volume preserving* C^2 *codimension one Anosov flow on a compact manifold* M *of dimension* $n > 3$. *If the distribution* E^{su} *is of class* Lip–, *then* $\{f_t\}$ *admits a global cross-section.*

Proof. The main idea is to prove that W^{ss} and W^{uu} are jointly integrable foliations. This means that, locally speaking, the projection from one center stable leaf to another along W^{uu}-leaves maps the strong stable leaves to strong stable leaves. Plante [**Pl1**] showed that when W^{ss} and W^{uu} are jointly integrable, then the Anosov flow admits a global cross-section, and we use his result.

Since the flow is volume preserving and $n > 3$, it follows from the C^1 section theorem of [**HPS**] that E^{uu} is a C^1 bundle. (Without loss of generality we may assume that it is also orientable.) Let Y and λ be as in §2; in our case, Y is C^1. Let $\{\phi_t\}$ be the flow of Y; clearly, ϕ_t is also C^1.

Let α be a 1-form on M defined by

$$\operatorname{Ker}(\alpha) = E^{su}, \quad \alpha(X) = 1,$$

where X is the Anosov vector field. Since E^{su} is of class Lip–, so is α. Moreover, α is invariant with respect to f_t, that is, $f_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$.

Now let v be an arbitrary vector in E^{ss}. There exist continuous functions $a_v : \mathbb{R} \to \mathbb{R}$ and $b_v : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$T\phi_t(v) = a_v(t)Y + b_v(t)X + Z_v(t),$$

where $Z_v(t) \in E^{ss}$. We will show that $b_v = 0$ for all $v \in E^{ss}$, by analyzing the behavior of $b_v(t)$ under iteration of the Anosov flow on the vector v. For that we need an estimate of the size of $b_v(t)$ for small t.

LEMMA 3.3. *Let b_v be as above. Then for every $0 < \theta < 1$ there exists a constant $C > 0$, independent of v, such that for all $|t| \le 1$,*

$$|b_v(t)| \le C|t|^\theta \|v\|.$$

Proof. Let $\bar{d}$ denote the distance function on TM (the tangent bundle of M) induced by some Riemannian metric on its tangent bundle. Since α is Lip–, for every $0 < \theta < 1$ and every $\epsilon > 0$ there exists a constant $A(\theta, \epsilon) > 0$ such that

$$|\alpha(u) - \alpha(w)| \le A(\theta, \epsilon)\bar{d}(u, w)^\theta, \tag{4}$$

for all $u, w \in TM$ for which $\bar{d}(u, w) \le \epsilon$. (Note that on noncompact spaces like TM, Hölderness is a condition valid only on a 'small scale'.) Also there exists an $\epsilon_0 > 0$ such that for every $v \in E^{ss}$ with norm ≤ 1 and every $|t| \le 1$,

$$\bar{d}(T\phi_t(v), v) \le \epsilon_0.$$

Let $0 < \theta < 1$ be arbitrary and set $B = A(\theta, \epsilon_0)$. Then for $|t| \le 1$ and $v \in E^{ss}$, $\|v\| \le 1$, we have

$$\begin{aligned} |b_v(t)| &= |\alpha(T\phi_t(v))| \\ &= |\alpha(T\phi_t(v)) - \alpha(v)| \\ &\le B\bar{d}(T\phi_t(v), v)^\theta \qquad (5) \\ &\le B\left|\int_0^t \left\| \frac{d}{ds}[T_x\phi_s(v)]\right\| ds\right|^\theta \qquad (6) \end{aligned}$$

$$\begin{aligned} &= B\left|\int_0^t \|(T_{\phi_s x}Y)(T_x\phi_s)(v)\|\,ds\right|^\theta && (7)\\ &\leq C|t|^\theta\|v\|^\theta, && (8)\end{aligned}$$

where $C = B(\sup_M \|TY\| \sup_{|s|\leq 1}\|T\phi_s\|)^\theta$. Inequality (5) holds because of (4), (6) follows from the definition of distance induced by a Riemannian metric (note that $s \mapsto T\phi_s(v)$, $0 \leq s \leq t$, is a path in TM connecting v and $T\phi_t(v)$), and (7) follows from the first variation equation applied to the C^1 vector field Y.

Now let $v \in E^{ss}$ be arbitrary. Then, since $b_v(t)$ is linear in v, we have

$$\begin{aligned} |b_v(t)| &= |\|v\|b_{v/\|v\|}(t)|\\ &\leq \|v\|C|t|^\theta\left\|\frac{v}{\|v\|}\right\|^\theta && (9)\\ &= C|t|^\theta\|v\|.\end{aligned}$$

Clearly, (9) follows from (8). This completes the proof of the lemma. □

Now set

$$\sigma(x,t,s) = \int_0^s \lambda(\phi_r x, t)\,dr.$$

Observe that for every fixed s, $\sigma(x,t,s) \to 0$ as $t \to -\infty$, uniformly in x.

The next step is to show that the flows f_t and ϕ_s satisfy the following commutation relation.

Lemma 3.4. *For all $x \in M$ and $t, s \in \mathbb{R}$,*

$$f_t\phi_s(x) = \phi_{\sigma(x,t,s)}f_t(x). \tag{10}$$

Proof. Since the foliation W^{uu} is invariant with respect to f_t, there is a function $\mu : M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that (10) holds with μ instead of σ. We have to show that $\mu = \sigma$.

Differentiate (10) (with μ instead of σ) with respect to s. We get

$$Tf_t(Y_{\phi_s x}) = \frac{\partial\mu}{\partial s}(x,t,s)Y_{f_t\phi_s(x)}.$$

Solving for $\partial\mu/\partial s$ and using the definition of λ, gives $(\partial\mu/\partial s)(x,t,s) = \lambda(\phi_s x, t)$, which directly implies $\mu = \sigma$. □

The question of how $b_v(t)$ behaves under iteration of the flow f_t on v is answered by the following lemma.

Lemma 3.5. *For every $v \in E^{ss}_x$ and $s, t \in \mathbb{R}$,*

$$b_v(s) = b_{Tf_t(v)}(\sigma(x,t,s)).$$

Proof. We have

$$b_v(s) = (\phi_s^*\alpha)(v)$$

$$
\begin{aligned}
&= (\phi_s^* f_t^* \alpha)(v) \\
&= [(f_t \phi_s)^* \alpha](v) \\
&= \alpha(T(f_t \phi_s)(v)) \\
&= \alpha(T(\phi_{\sigma(x,t,s)} f_t)(v)) && (11) \\
&= \alpha\left(T\phi_{\sigma(x,t,s)} T f_t(v) + \frac{\partial \sigma}{\partial x}(v) Y\right) && (12) \\
&= \alpha(T\phi_{\sigma(x,t,s)} T f_t(v)) && (13) \\
&= b_{Tf_t(v)}(\sigma(x,t,s)),
\end{aligned}
$$

as desired. Equality (11) follows from Lemma 3.4, (12) follows by the chain rule, and (13) holds because $\alpha(Y) = 0$. ☐

Set $\nu = \sup_{x \in M} \|T_x f_1|_{E^{ss}}\|$. Let $s \neq 0$, $|s| \leq 1$, and $v \in E^{ss}$ be arbitrary but fixed. We will show that $b_v(s) = 0$.

By Corollary 2.2 it follows that if $|s| \leq 1$, then

$$
\begin{aligned}
|\sigma(x,-t,s)| &= \left| \int_0^s \lambda(\phi_r x, -t)\, dr \right| \\
&\leq \left| \int_0^s e^{\ell r} \lambda(x,-t)\, dr \right| \\
&= e^{\ell} |s| \lambda(x,-t). && (14)
\end{aligned}
$$

Since $\lambda(x,-t)$ tends to zero exponentially and $n > 3$, we can choose θ sufficiently close to one so that

$$
\nu^{(n-3)t} \sigma(x,-t,s)^{\theta-1} \to 0, \tag{15}
$$

as $t \to \infty$ uniformly in x. Let $C' = Ce^{\ell}$. Then we have

$$
\begin{aligned}
|b_v(s)| &= |b_{Tf_{-t}(v)}(\sigma(x,t,s))| \\
&\leq C|\sigma(x,-t,s)|^{\theta} \|Tf_{-t}(v)\| && (16) \\
&= C|\sigma(x,-t,s)| \|Tf_{-t}(v)\| |\sigma(x,-t,s)|^{\theta-1} \\
&\leq C'|s|\lambda(x,-t) \|Tf_{-t}(v)\| |\sigma(x,-t,s)|^{\theta-1} && (17) \\
&= C'|s| \|Tf_{-t}(Y_x \wedge v)\| |\sigma(x,-t,s)|^{\theta-1} \\
&\leq C'|s| \|v\| \cdot \nu^{(n-3)t} |\sigma(x,-t,s)|^{\theta-1} && (18) \\
&\to 0,
\end{aligned}
$$

as $t \to \infty$. Inequality (16) follows from Lemma 3.3 (clearly, we may assume t is so large that $|\sigma(x,-t,s)| \leq 1$), (17) follows from (14), while (18) is a consequence of Lemma 3.1. The expression in (18) tends to zero by (15). Note that the hypothesis $n > 3$ is strongly used here.

Therefore, $b_v(s) = 0$ for every $v \in E^{ss}$ and $|s| \leq 1$. It is easy to verify that

$$
b_v(t+s) = b_v(t) + b_{Z_v(t)}(s),
$$

which immediately implies that $b_v(t) = 0$ for *every* $t \in \mathbb{R}$ and every $v \in E^{ss}$, i.e. $T\phi_t(v)$ has no component in the X direction.

To complete the proof of Theorem 3.2, we need to show that the foliations W^{ss} and W^{uu} are jointly integrable.

Let U be an open subset of M and let L_0 and L_1 be the plaques lying in U of two leaves of the foliation W^{cs}. Choose U so small that the projection $P : L_0 \to L_1$ along the leaves of W^{uu} is well defined. There is a continuous function $\tau : L_0 \to \mathbb{R}$ such that for all $x \in L_0$,

$$P(x) = \phi_{\tau(x)}(x).$$

Since W^{uu} is C^1, so are τ (by the implicit function theorem) and P.

Now let $c : J \to L_0$ be a C^1 curve lying in a single strong stable leaf in L_0 (J is some open interval in $\mathbb{R}$). Then by the chain rule

$$\frac{d}{ds}[P(c(s))] = T\phi_{\tau(c(s))}(\dot{c}(s)) + d\tau(\dot{c}(s))Y.$$

Therefore, $TP(\dot{c}(s)) = (d/ds)[P(c(s))]$ has no component in the X direction. Since $TP(\dot{c}(s))$ also belongs to E^{cs} for all $s \in J$, it follows that, in fact, $TP(\dot{c}(s))$ belongs to E^{ss} for all s. Since the path c in W^{ss} was arbitrary, we have proved

$$P(W^{ss}(x) \cap L_0) \subset W^{ss}(P(x)) \cap L_1,$$

for all $x \in L_0$. Therefore W^{ss} and W^{uu} are jointly integrable and the proof of Theorem 3.2 is complete. □

4. *Synchronization*

In this section we show that it is possible to suitably reparametrize certain types of Anosov flows so that their properties become ‘synchronized’ in the sense described below. The results are similar to some conclusions of Marcus [**Ma1**], [**Ma2**], Margulis [**Mg**] and Ghys [**Gh1**], [**Gh2**], but our methods differ from theirs.

First we recall some basic facts. Suppose $\{f_t'\}$ is a reparametrization of an Anosov flow $\{f_t\}$ on M obtained by multiplying the generating vector field X by a C^1 function v: $X' = vX$. Then f_t' is also an Anosov flow whose orbits coincide with the orbits of the original flow. In particular, f_t' admits a cross-section if and only if f_t does. Furthermore, the center stable distributions of f_t' and f_t are identical, and the same is true for the center unstable ones. It is, in fact, possible to show that every vector in the strong stable distribution $E^{ss'}$ of f_t' is of the form $v + \xi(v)X$ for some $v \in E^{ss}$, where ξ is some 1-form on E^{ss}. (An analogous statement is true for the strong unstable distributions.) A word of caution is needed here: reparametrization does *not* preserve the strong distributions unless v is a constant function.

If f_t' is a reparametrization of a flow (not necessarily Anosov) f_t by a function v, then there exists a function $\varrho : M \times \mathbb{R} \to \mathbb{R}$ such that

$$f_t'(x) = f_{\varrho(x,t)}(x)$$

for all $x \in M$ and $t \in \mathbb{R}$. We state the following lemma and omit its straightforward proof.

LEMMA 4.1.

(a) *For all x and t.*

$$\varrho(x,t) = \int_0^t v(f'_s x)\,ds.$$

(b) $\varrho(x, t+s) = \varrho(x,t) + \varrho(f'_t x, s) \quad (x \in M,\ t \in \mathbb{R})$.

(c) *For every $x \in M$ and $w \in T_x M$,*

$$T_x f'_t(w) = T_x f_{\varrho(x,t)}(w) + \frac{\partial \varrho}{\partial x}(w) X.$$

Let $\{f_t\}$ be a C^r $(r \geq 2)$ Anosov flow on M. We make the following standing hypothesis for this section:

- $\{f_t\}$ is transitive (i.e. there is a dense orbit);
- $E^{cs} = E^{ss} \oplus E^c$ is of class C^1;
- E^{uu} is orientable (but not necessarily one-dimensional).

Let F be a C^∞ sub-bundle of TM complementary to E^{cs}, i.e. $\dim F = \dim E^{uu} = k$ and $TM = E^{cs} \oplus F$, and choose a smooth Riemann structure $\mathcal{R}$ on M. Since F is orientable, it has a C^∞ volume k-form ω_F. Define a k-form ω on the whole TM by extending ω_F as follows:

$$\operatorname{Ker}(\omega) = E^{cs}, \quad \omega|_F = \omega_F.$$

The first part of the definition just means that $i_V \omega = 0$, for every vector $V \in E^{cs}$, where i_V denotes inner multiplication by V. (Good references for elements of the calculus of differential forms are [**Wa**] and [**GHL**].)

Our next aim is to modify the existing Riemann structure $\mathcal{R}$ on M, so that with respect to the new structure $\mathcal{R}_*$, ω behaves nicely when pulled back by the Anosov flow.

Let $p : E^{uu} \to F$ be the bundle isomorphism given by an orthogonal projection relative to $\mathcal{R}$. Define a new Riemann structure $\mathcal{R}_*$ on M by declaring the following:

(1) with respect to $\mathcal{R}_*$, E^{uu} is orthogonal to E^{cs};
(2) $\mathcal{R}_*$ and $\mathcal{R}$ coincide on E^{cs};
(3) $p : (E^{uu}, \mathcal{R}_*) \to (F, \mathcal{R})$ is an isometry.

We call $\mathcal{R}_*$ the Riemann structure *adapted to ω and E^{uu}*. The important point in its definition is (3), as we shall see below.

Finally, let $\lambda(x,t)$ be the determinant of $T_x f_t|_{E^{uu}}$ relative to $\mathcal{R}_*$. Then we have the following result.

THEOREM 4.2.

(a) *For all $x \in M$ and $t \in \mathbb{R}$,*

$$(f_t^* \omega)_x = \lambda(x,t)\omega_x.$$

(b) *ω is C^1 and $d\omega = \eta \wedge \omega$ for some continuous 1-form η.*

(c) *Let $u = \eta(X)$; call it the 'u-function' corresponding to ω and X. Then u is C^1 and*

$$u(f_t x) = \left.\frac{d}{ds}\right|_t \log \lambda(x,s).$$

Proof. (a) First note the following. If C is an $\mathcal{R}_*$-unit cube in E^{uu}, then, by construction, $p(C)$ is an $\mathcal{R}$-unit cube in F. Furthermore, if $C = Y_1 \wedge \cdots \wedge Y_k$, then $p(Y_j) - Y_j \in E^{cs}$, so $\omega(C) = \omega(p(C))$. Since $\omega|_F$ was an $\mathcal{R}$-volume form for F, we have

$$\omega(C) = \omega(p(C)) \equiv 1.$$

Now let $U \subset M$ be a small open set over which E^{uu} is trivial and let $x \mapsto C_x$ be a continuous family of $\mathcal{R}_*$-unit cubes in E^{uu}, for $x \in U$. Then for small t, $f_{t*}(C) = \lambda(x,t)C_{f_t x}$. Therefore,

$$\begin{aligned} (f_t^*\omega)_x(C_x) &= \omega(f_{t*}(C_x)) \\ &= \lambda(x,t)\omega_{f_t x}(C_{f_t x}) \\ &= \lambda(x,t) \\ &= \lambda(x,t)\omega_x(C_x), \end{aligned}$$

hence, $(f_t^*\omega)_x = \lambda(x,t)\omega_x$, for small t. But by the cocycle property of λ, this identity extends over all $(x,t) \in M \times \mathbb{R}$.

(b) By our standing assumption in this section that E^{cs} is C^1, and since F is C^∞, it follows that ω is C^1. The second part of (b) is just a restatement of the Frobenius theorem for the integrable distribution E^{cs}.

(c) To show that u is C^1, let $Z_1, \ldots, Z_k$ be a C^∞ $\mathcal{R}$-orthonormal frame in $F|_U$, which trivializes F over some small open set U, and such that $\omega(Z_1, \ldots, Z_k) \equiv 1$. Then we have

$$\begin{aligned} u &= \eta(X) \\ &= \eta(X)\omega(Z_1, \ldots, Z_k) \\ &= (\eta \wedge \omega)(X, Z_1, \ldots, Z_k) && (19) \\ &= d\omega(X, Z_1, \ldots, Z_k) && (20) \\ &= X(\omega(Z_1, \ldots, Z_k)) + \sum_{i=1}^{k}(-1)^i Z_i(\omega(X, Z_1, \ldots, \widehat{Z_i}, \ldots, Z_k)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([Z_i, Z_j], Z_0, \ldots, \widehat{Z_i}, \ldots, \widehat{Z_j}, \ldots, Z_k) && (21) \\ &= \sum_{i<j}(-1)^{i+j}\omega([Z_i, Z_j], Z_0, \ldots, \widehat{Z_i}, \ldots, \widehat{Z_j}, \ldots, Z_k), && (22) \end{aligned}$$

where $Z_0 = X$ and the hat denotes omission. The expression in (22) is of class C^1 because both ω and $[Z_i, Z_j]$ are C^1, for all i, j. Identity (19) holds because $i_X\omega = 0$, (20) follows from the Frobenius theorem ($d\omega = \eta \wedge \omega$), and (21) can be found in [**Wa**, Proposition 2.25(f), p. 70]. Finally, (22) holds because $\omega(Z_1, \ldots, Z_k)$ is identically equal to one (by definition), so its X derivative is zero, and $Z_i(\omega(X, Z_1, \ldots, \widehat{Z_i}, \ldots, Z_k)) = Z_i(0) = 0$.

To show the second part of (c), consider (20) and recall that the Lie derivative (with respect to X), L_X, can be written as $L_X = di_X + i_X d$. Since $di_X\omega = d0 = 0$, and (20) is equal to $(i_X d\omega)(Z_1, \ldots, Z_k)$, we have

$$u = (L_X\omega)(Z_1, \ldots, Z_k)$$

$$\begin{aligned} &= \left.\frac{d}{dt}\right|_0 (f_t^*\omega)(Z_1, \ldots, Z_k) \qquad (23) \\ &= \left.\frac{d}{dt}\right|_0 \lambda(x, t). \end{aligned}$$

Identity (23) is an alternative definition of the Lie derivative of a differential form. The proof can now be completed in exactly the same way as in part (b) of Theorem 2.1. □

Observe that u measures the rate of expansion of the flow along E^{uu}. If $u(x) < 0$ for some x, then in a neighborhood of x in M, $Tf_t|_{E^{uu}}$ might not really be expanding for small t. To avoid this unnatural and temporary behavior, we show that it is possible to modify the Riemann structure $\mathcal{R}_*$ (and therefore modify λ) in a continuous fashion to obtain $u > 0$. Thus, after this modification, f_t becomes 'immediately expanding' on E^{uu}.

LEMMA 4.3. *Let $\{f_t\}$ be as above. Then there exists a continuous Riemann structure on M with respect to which $u > 0$.*

Proof. Recall that $\lambda(x, t) \geq Ke^{ct}$, for all $x \in M$, $t > 0$ and some $K, c > 0$. If $K \geq 1$, then by Theorem 4.2(c) (see also Theorem 2.1(c) for a similar calculation),

$$\frac{1}{\tau}\int_0^\tau u(f_t x)\, dt \geq \frac{1}{\tau}\log K + c \geq c > 0,$$

for every $x \in M$ and $\tau > 0$, so by letting $\tau \to 0+$, we obtain $u(x) > 0$, and we are done.

If $K < 1$, choose $\tau_0 > 0$ so that $c_0 = (\log K)/\tau_0 + c > 0$. Again by Theorem 4.2(c), if a periodic point x has period $\tau \geq \tau_0$, then

$$\frac{1}{\tau}\int_0^\tau u(f_t x)\, dt \geq c_0 > 0.$$

Let P be the union of all periodic points of $\{f_t\}$ whose period is longer than τ_0. Then there are only finitely many periodic orbits of $\{f_t\}$ which are *not* in P (for details see [**PM**, p. 100]), so by transitivity of the flow, P is dense in M. A standard argument (see, for instance, [**Gh3**, Lemma 2.4]) shows that every f_t-invariant Borel probability measure on M can be approximated by convex combinations of invariant probabilities concentrated on periodic orbits *in* P, so the previous inequality implies

$$\int_M u\, d\nu \geq c_0 > 0,$$

for every invariant Borel probability measure ν on M. Approximate u by a C^∞ function w such that $\delta = \sup_M |u - w| < c_0/4$. Let $u_0 = w - \delta$. Then

$$u_0 = (w - u - \delta) + u \leq u,$$

and for any invariant probability measure ν on M

$$\begin{aligned} \int_M u_0\, d\nu &= \int_M (u_0 - u)\, d\nu + \int_M u\, d\nu \\ &\geq -2\delta + c_0 \\ &\geq \tfrac{1}{2}c_0. \end{aligned}$$

The following 'sublemma' due to Ghys [**Gh3**] makes use of this property of u_0.

SUBLEMMA. *There exists a C^∞ function $v : M \to \mathbb{R}$ such that $X(v) < u_0$.*

We omit the proof which can be found in [**Gh3**, Lemma 2.5].

Continuing with the proof of Lemma 4.3, consider the 1-form

$$\bar{\omega} = e^{-v}\omega.$$

It is of class C^1 and defines the same plane field as ω, namely E^{cs}. It is easy to see that

$$d\bar{\omega} = (\eta - dv) \wedge \bar{\omega}.$$

Furthermore, $(\eta - dv)(X) = u - X(v) \geq u_0 - X(v) > 0$. Let $\bar{\eta} = \eta - dv$ and $\bar{u} = \bar{\eta}(X)$. Note that $\bar{\eta}(X)$ is the u-function corresponding to $\bar{\omega}$ and X. We have $\bar{u}(x) > 0$ for all $x \in M$ and $\bar{\eta}$ satisfies part (c) of Theorem 4.2 if $\lambda(x,t)$ is taken relative to the Riemann structure adapted to $\bar{\omega}$ and E^{uu}. This completes the proof of Lemma 4.3. □

Thus, without loss of generality we may assume that, with respect to some continuous Riemann structure on M, the u-function corresponding to ω and X satisfies $u > 0$.

Since $u > 0$, the vector field

$$X' = \frac{1}{u}X$$

is well defined. Let $\{f_t'\}$ denote its flow. Then f_t' is an Anosov flow and its center distributions coincide with the corresponding center distributions of f_t. Clearly, f_t' is also transitive. Let $E^{uu'}$ be the strong unstable distribution of f_t'. As remarked above, there is a 1-form ξ such that every vector w in $E^{uu'}$ can be written in the form $w = v + \xi(v)X$ for some $v \in E^{uu}$. Therefore, $E^{uu'}$ is an orientable bundle. Having verified the hypothesis at the beginning of the section, the synchronization procedure used above will now be applied to the flow $\{f_t'\}$. Observe that f_t' is only of class C^1; however, this is not a problem simply because we do not need 'C^r-ness' of the flow any more (we originally needed it with $r \geq 2$ because the invariant section theorem of [**HPS**] requires that the bundle map be at least as differentiable as the invariant distribution obtained by it). So we proceed in the following manner.

First we perform the 'adaptation' (as described above) of the Riemann structure to ω and $E^{uu'}$, and let $\lambda'(x,t)$ be the determinant of $T_x f_t'$ on $E^{uu'}$ with respect to the adapted metric. Since the center stable distribution of f_t' coincides with the one for the original flow, we can 'recycle' ω (which is defined by E^{cs}); hence we can also 'recycle' η. Set $u' = \eta(X')$. This is the u-function corresponding to ω and X'. Therefore, the formula in Theorem 4.2(c) remains valid in this new setting; namely,

$$\left.\frac{d}{ds}\right|_t \log \lambda'(x,s) = \eta(X') = u(f_t'x).$$

However,

$$u' = \eta\left(\frac{1}{u}X\right) = 1,$$

and hence

$$\lambda'(x,t) \equiv e^t.$$

Thus we have proved the following result.

THEOREM 4.4. *Let* $\{f_t\}$ *be a transitive* C^r $(r \geq 2)$ *Anosov flow on a compact manifold* M*, such that its center stable distribution is of class* C^1 *and its strong unstable distribution is orientable. Then there exists a continuous Riemann structure on* M *and a* C^1 *reparametrization* $\{f'_t\}$ *of* $\{f_t\}$ *such that the determinant of* f'_t *on its strong unstable distribution is identically equal to* e^t.

Moreover, if the center stable bundle of the original flow is of class C^s *for some* $s \geq 1$*, then the new flow is of class* $C^{\min(r,s)}$*, with noninteger values of* r *and* s *allowed.*

We will call $\{f'_t\}$ the *synchronization* of the flow $\{f_t\}$.

We suppose that the reader is now curious to see what are some possible advantages of the synchronized flow over the original flow, so we refer to the next two sections where we focus on codimension one flows for which these advantages are most visible.

5. *Some applications of synchronization*

We will say that a Lie group G acts *locally freely* on a manifold M if the isotropy group (or stabilizer) of the action at every point is a discrete subgroup of G. In that case, the orbits of the action form a foliation of M of dimension $\dim G$.

Denote by G the Lie group of orientation-preserving affine transformations of the real line. Then we have the following.

THEOREM 5.1. *If a compact manifold* M *admits a* C^2 *transitive codimension one Anosov flow whose strong unstable bundle is orientable, then* G *acts locally freely on* M*. This action (i.e. the map* $G \times M \to M$*) is of class* C^1.

Proof. Let $\mathbf{g}$ be the Lie algebra of G. It is well known (see, for instance, [**Wa**]) that $\mathbf{g}$ has two generators, A and B, which satisfy the relation

$$[A, B] = -B.$$

Let X be the Anosov field of the synchronized flow $\{f_t\}$ and Y a section of its strong unstable bundle such that $Tf_t(Y) = e^t Y$. (This can be done by Theorem 4.4.) Then, clearly, $[X, Y] = -Y$, which implies that G acts on M. The orbits of this action are the leaves of the center stable foliation of the synchronized flow. Let $x \in M$ be arbitrary, and denote by S_x the stabilizer of the action at x, that is, the set of all elements of G which have x as a fixed point. If x is a periodic point of the Anosov flow, then S_x is free cyclic, otherwise S_x is trivial. Therefore, the action of G on M is locally free. It is not difficult to see that it is also of class C^1. □

It is easy to see that the synchronization of a volume-preserving flow is volume preserving: if the flow of X preserves a volume form Ω, and $X' = (1/u)X$ is the synchronized vector field, then the flow of X' preserves the volume form $\Omega' = u\Omega$:

$$\begin{aligned} L_{X'}\Omega' &= (di_{X'} + i_{X'}d)\Omega' \\ &= di_{X'}\Omega' \\ &= di_X\left(\frac{1}{u}\Omega'\right) \\ &= L_X\Omega \\ &= 0. \end{aligned}$$

For simplicity of notation, denote the synchronized flow of a volume-preserving codimension one Anosov flow by $\{f_t\}$, the vector field tangent to it by X and assume it preserves a C^1 volume form Ω. Let Y be a continuous unit vector field in the strong unstable bundle of $\{f_t\}$ which is the projection to E^{uu} of some C^∞ vector field Z everywhere transverse to E^{cs} (cf. the previous section). Define 1-forms α and ω, and an $(n-2)$-form θ by

- $\mathrm{Ker}(\alpha) = E^{su}$, $\alpha(X) = 1$;
- $\mathrm{Ker}(\omega) = E^{cs}$, $\omega(Y) = 1$;
- $\theta = i_X i_Y \Omega$.

Then ω is of class C^1, while α and θ are only continuous. However, the nullspace of θ, i.e. E^{cu}, is an integrable distribution so by Hartman's version of the Frobenius theorem $d\theta$ exists in the Stokes sense. Furthermore, we have the following result which we will use in the next section.

THEOREM 5.2.
(a) *$d\omega = \alpha \wedge \omega$.*
(b) *$d\theta = -\alpha \wedge \theta$.*
(c) *α is closed on the leaves of W^{cs} and W^{cu}.*

Proof. (a) By the Frobenius theorem there is a continuous form η such that $d\omega = \eta \wedge \omega$. Since $\det Tf_t$ on E^{uu} is identically e^t, we have $f_t^*\omega = e^t\omega$ for all t. It follows that $f_t^*(d\omega) = df_t^*\omega = d(e^t\omega) = e^t d\omega$, hence

$$\begin{aligned}(f_t^*\eta) \wedge \omega &= e^{-t} f_t^*(\eta \wedge \omega)\\ &= e^{-t} f_t^*(d\omega)\\ &= e^{-t}e^t \eta \wedge \omega\\ &= \eta \wedge \omega.\end{aligned}$$

Elementary calculus of differential forms implies that there exists a continuous function $k : \mathbb{R} \times M \to \mathbb{R}$ such that

$$f_t^*\eta - \eta = k(t, x)\omega. \tag{24}$$

Evaluate both sides of (24) at an arbitrary vector $v \in E^{ss}$. Since $\omega(v) = 0$, we obtain

$$\begin{aligned}|\eta(v)| &= |(f_t^*\eta)(v)|\\ &= |\eta(Tf_t(v))|\\ &\le \|\eta\|_\infty \|Tf_t(v)\|\\ &\to 0,\end{aligned}$$

as $t \to \infty$. Since we know from before that $\eta(X) = 1$ (see §4), it follows that the restrictions of α and η to the distribution E^{cs} are identical. Thus, there is a function $h : M \to \mathbb{R}$ such that $\eta = \alpha + h\omega$, which implies

$$\eta \wedge \omega = \alpha \wedge \omega,$$

as desired. Note that the proof does not use volume preservation.

(b) By the Frobenius theorem there exists a continuous 1-form β such that $d\theta = \beta \wedge \theta$. We will show that the restrictions of β and $-\alpha$ to E^{cu} coincide.

Observe that the determinant of Tf_t on E^{ss} is identically e^{-t}. So just as we showed that $f_t^*\omega = e^t\omega$, we can in exactly the same fashion show that

$$f_t^*\theta = \det Tf_t|_{E^{ss}} = e^{-t}\theta,$$

i.e. $L_X\theta = -\theta$. Since $L_X = di_X + i_X d$ and $i_X\theta = 0$, we have

$$\begin{aligned} -\theta &= i_X\, d\theta \\ &= i_X(\beta \wedge \theta) \\ &= \beta(X)\theta. \end{aligned}$$

Thus $\beta(X) = -1$. Following the procedure in part (a), it is easy to show that $\beta(Y) = 0$. Hence $-\beta$ and α are identical as forms on the leaves of W^{cu} and therefore $d\theta = -\alpha \wedge \theta$, as desired.

(c) Part (a) implies (see, for instance, [**HH**]) that the integral of α over any loop contained in a leaf of W^{cs} equals the logarithm of the linear part of the holonomy of the foliation W^{cs}. So if D is a disk contained in a leaf of W^{cs}, then $\int_{\partial D} \alpha = 0$, since trivial loops (such as ∂D) carry no holonomy. Thus the Stokes differential of α on each leaf of W^{cs} is zero.

Similarly, the integral of $-\alpha$ over a loop contained in a leaf of W^{cu} is equal to the logarithm of the absolute value of the determinant of the holonomy of the foliation W^{cu}. The same argument as above then shows that $-\alpha$ is closed in the Stokes sense on every leaf of W^{cu}. □

More applications of the method of synchronization are given in the next section.

6. *Invariant forms and the Godbillon–Vey class of the center stable foliation*

In this section we consider bounded forms invariant with respect to a codimension one Anosov flow. We showed in [**Si1**] and [**Si2**] (see also [**Gh3**]) that such forms of degree two must vanish everywhere; here we extend that result and apply it to a discussion of the Godbillon–Vey class of the center stable foliation.

THEOREM 6.1. *Every bounded k-form on a compact manifold M of dimension $n > 3$ which is invariant with respect to some codimension one Anosov flow on M vanishes if $2 \leq k \leq n-2$.*

Proof. Let β be a bounded f_t-invariant form of degree k on M, with $2 \leq k \leq n-2$, where f_t is some codimension one Anosov flow on M. Since the proof in the case $k = 2$ will be obvious from what follows, we will assume $k > 2$.

Choose a continuous Riemann structure on M, relative to which the bundles E^c, E^{ss} and E^{uu} are orthogonal. As before, let X be the Anosov vector field and let Y be a continuous section of E^{uu}. Without loss we may assume X and Y have unit length. Let $x \in M$ be an arbitrary point and at x, choose an arbitrary orthonormal basis of E^{ss}: $v_1, \ldots, v_{n-2}$. It suffices to show that

$$\beta(X, Y, v_1, \ldots, v_{k-2}) = 0 \quad \text{and} \quad \beta(Y, v_1, \ldots, v_{k-1}) = 0.$$

Let $C = X \wedge Y \wedge v_1 \wedge \cdots \wedge v_{n-2}$ be the corresponding unit cube in $T_x M$. Denote by $\Delta(x,t)$ the determinant of $T_x f_t$. Thus $\Delta(x,t) = \| f_{t*}(C)\|$. As shown by Plante in [**Pl1**], Egoroff's theorem implies that the set of points x for which $\Delta(x,t)$ tends to ∞ as $|t| \to \infty$ has Lebesgue measure zero. Therefore for a.e. $x \in M$, there is a sequence (t_i) converging to ∞ such that $\Delta(x,t_i)$ stays bounded as $i \to \infty$. Call such a sequence (t_i) a *good sequence* and the corresponding points (for which there is a good sequence), *good points*. Consider the following two cases.

Case 1. Let $\nu > 1$ be such that $\|Tf_{-t}(v)\| \geq \nu^t \|v\|$, for large t and $v \in E^{ss}$. Set $B = \|\beta\|_\infty$. If x is a good point, then

$$
\begin{aligned}
|\beta(X, Y, v_1, \ldots, v_{k-2})| &= |(f_{-t}^*\beta)(X, Y, v_1, \ldots, v_{k-2})| \\
&= |\beta(f_{-t*}(X \wedge Y \wedge v_1 \wedge \cdots \wedge v_{k-2}))| \\
&\leq B\|f_{-t*}(X \wedge Y \wedge v_1 \wedge \cdots \wedge v_{k-2})\| \\
&= B\frac{\|f_{-t*}(C)\|}{\|f_{-t*}(v_{k-1} \wedge \cdots \wedge v_{n-2})\|} \\
&\leq B\Delta(x,t)\nu^{-(n-k)t} \\
&\to 0,
\end{aligned}
$$

where $t \to \infty$ along a good sequence. Note that $n - k \geq 2$.

Case 2. Similarly, if x is a good point, we have

$$
\begin{aligned}
|\beta(Y, v_1, \ldots, v_{k-1})| &= |(f_{-t}^*\beta)(Y, v_1, \ldots, v_{k-1})| \\
&= |\beta(f_{-t*}(Y \wedge v_1 \wedge \cdots \wedge v_{k-1}))| \\
&\leq B\frac{\|f_{-t*}(C)\|}{\|f_{-t*}(X \wedge v_k \wedge \cdots \wedge v_{n-2})\|} \\
&\leq B\Delta(x,t)\nu^{-(n-k-1)t} \\
&\to 0,
\end{aligned}
$$

as $t \to \infty$ along a good sequence. Observe that $n - k - 1 \geq 1$.

Since the set of good points has full measure in M, the proof is now complete. □

Let us apply the previous theorem to a calculation of the Godbillon–Vey class of the center stable foliation of a codimension one Anosov flow. Recall that for a C^2 codimension one foliation $\mathcal{F}$ whose tangent bundle is the nullspace of a C^2 1-form ω, the *Godbillon–Vey class of* $\mathcal{F}$, $GV(\mathcal{F})$, is a class in the third de Rham cohomology space $H^3(M,\mathbb{R})$ of the underlying manifold M, defined in the following way. By the Frobenius theorem there exists a C^1 form η such that $d\omega = \eta \wedge \omega$. It can be shown (see [**To**]) that the de Rham cohomology class of the 3-form $\eta \wedge d\eta$ does not depend on the choice of η, nor on the choice of ω. So we can define

$$GV(\mathcal{F}) = \{\eta \wedge d\eta\},$$

where the curly braces signify de Rham cohomology class.

Remark. The definition of the Godbillon–Vey class was extended by Hurder and Katok in [**HK**] to codimension one foliations of class $C^{1+\epsilon}$ on 3-manifolds where $\epsilon > 1/2$. It follows from a theorem of Rademacher that the Godbillon–Vey class is defined when the codimension one foliation is $C^{1+\mathrm{Lip}}$. However, despite the general feeling of experts that the Godbillon–Vey class is definable in higher dimensions (>3) for codimension one foliations of class $C^{1+\epsilon}$ with $\epsilon > 1/2$, I am still not aware of any written proof of that.

Now consider the center stable foliation of a codimension one Anosov flow in dimension > 4 and assume that it is possible to define its Godbillon–Vey class, GV. Consider the synchronization f_t of the original flow and the Godbillon–Vey class of its center stable foliation which, clearly, coincides with GV. Theorem 5.2 suggests that, since $d\omega = \alpha \wedge \omega$, and $GV = \{\alpha \wedge d\alpha\}$ (where it remains to make sense out of $d\alpha$ in the case when α is just Hölder), there is a representative of GV which is invariant with respect to f_t. Theorem 6.1 then implies that $GV = \mathbf{0}$. Because of this we make the following conjecture.

Conjecture. If the Godbillon–Vey class of the center stable foliation of a codimension one Anosov flow in dimension > 4 is definable, then it automatically vanishes.

The following theorem gives a positive answer to the conjecture provided that W^{cs} is differentiable enough. Part (a) of the theorem appears in a weaker form in the paper [**Gh3**] with incomplete proof. (Namely, in [**Gh3**] the following result from [**Ve**] is used, the proof of which is incorrect: the lift of the center stable foliation of any codimension one Anosov flow to the universal covering space is given by a C^1 submersion. Étienne Ghys has informed me that he has an alternative unpublished proof of his result.)

THEOREM 6.2. *Let $\{f_t\}$ be a C^2 volume preserving codimension one Anosov flow on a compact Riemannian manifold M of dimension $n > 3$. If the center stable distribution E^{cs} of the flow is of class $C^{1+\mathrm{Lip}-}$, then:*
(a) *the flow admits a global cross-section;*
(b) *the Godbillon–Vey class of W^{cs} is zero.*

Proof. (a) First note that the main obstacle in this proof is that the strong unstable bundle of the synchronization of $\{f_t\}$ does not have to be of class C^1, despite volume preservation. The reason is simple: the synchronized flow is not necessarily of class C^2, so the C^1 section theorem does not apply. However, we claim that the strong unstable bundle of the synchronized flow is Lip–.

To see this, assume for simplicity of notation that the original flow has been synchronized; call it $\{f_t\}$ and let the corresponding splitting be $TM = E^c \oplus E^{ss} \oplus E^{uu}$. Since the u-function used in reparametrization is as smooth as W^{cs} (see Theorem 4.4), it follows that Tf_t is Lip–. Let the Lipschitz constant of f_{-1} be μ. Clearly, $\mu = \|Tf_{-1}|_{E^{ss}}\| > 1$. Note that the norm of Tf_{-1} on E^{uu} is $1/e$, because the E^{uu}-determinant has been synchronized. Since $\dim M > 3$, we have that $\mu/e < 1$, and, in particular, $\mu^\theta/e < 1$, for every $\theta < 1$. The Hölder section theorem (see [**Sh**, Theorem 5.18(c)]) now implies that E^{uu} is θ-Hölder for all $\theta < 1$, as desired.

By Theorem 5.2, we may assume

$$d\omega = \alpha \wedge \omega, \tag{25}$$

where ω and α have the same meaning as before. Note that (25) alone does not yet imply that α is of class Lip–, which is what we would like to show. However, since ω is $C^{1+\text{Lip}-}$, there exists a Lip– 1-form η such that $d\omega$ is also equal to $\eta \wedge \omega$. Elementary calculus of differential forms implies that $\eta - \alpha$ is a multiple of ω, i.e. there exists a continuous function $h : M \to \mathbb{R}$ such that

$$\eta = \alpha + h\omega. \tag{26}$$

Evaluate both sides of (26) at Y, a nonsingular Lip– section of E^{uu} such that $\omega(Y) = 1$. Then since $\alpha(Y) = 0$, we have

$$\eta(Y) = h.$$

Therefore, h is of class Lip–, hence so is α. By Theorem 3.2, it follows that $\{f_t\}$ admits a global cross-section.

(b) By part (a) and Theorem 3.2 we actually have that W^{ss} and W^{uu} are jointly integrable. This, by [**Pl1**], implies that E^{su} is integrable. Therefore, by [**Ha**], $d\alpha$ exists in the Stokes sense, and $\alpha \wedge d\alpha = 0$. It is easily seen that this implies $d\alpha = 0$. Thus

$$GV(W^{cs}) = \{\alpha \wedge d\alpha\} = \mathbf{0},$$

as desired. □

Remark. Hurder and Katok showed that if $\dim M = 3$, then the center stable foliation is, indeed, of the differentiability class required in Theorem 6.2 (in fact, they showed even more; see [**HK**]). It is an open question as to under which conditions the same statement is true in higher dimensions.

Note that a slight modification of the proof of Theorem 6.2 gives us the following simple lemma on differential forms.

LEMMA 6.3. *Suppose α, η and ω are 1-forms on a manifold M, and Y is a C^1 vector field on M. If η and ω are of class C^1, $\omega(Y) > 0$, $\alpha(Y) = 0$ and $\alpha \wedge \omega = \eta \wedge \omega$, then α is also of class C^1.*

COROLLARY 6.4. *If $\{f_t\}$ is a C^2 volume preserving codimension one Anosov flow on a compact Riemannian manifold M of dimension $n > 3$ and its center stable foliation W^{cs} is of class C^2, then both the strong stable and strong unstable bundle of the synchronization of $\{f_t\}$ are of class C^1.*

Proof. It follows from Lemma 6.3 that the E^{su}-bundle (call it F) of the synchronized flow $\{f'_t\}$ as well as its strong unstable bundle (call it F^{uu}) are of class C^1. Let us show that the strong stable bundle, F^{ss} of the synchronized flow is also C^1. We will use the C^1 section theorem of [**HP**]. More specifically, we closely follow the proof of Theorem 6.3 of [**HP**].

Approximate in the C^0 sense F^{ss} by a C^1 sub-bundle F^0 of F. For each $x \in M$ let L_x be the space of linear maps $F^0 \to F^{uu}$ with norm ≤ 1. We seek F^{ss}_x as the graph of an element of L_x, that is, we are looking for a section of the bundle $L \to M$ invariant relative to Th, where $h = f'_{-1}$.

Let $\Gamma : L \to L$ be the graph transform induced by $Th : F^0 \oplus F^{uu} \to F^0 \oplus F^{uu}$. If the matrix of Th relative to the splitting $F^0 \oplus F^{uu}$ is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then Γ is defined by the formula

$$\Gamma(\sigma) = (C + D\sigma) \cdot (A + B\sigma)^{-1},$$

where σ is a section of $L \to M$. Let $\epsilon > 0$ be as small as we want. Then we can choose F^0 so close to F^{ss} that the Lipschitz constant of Γ, $L(\Gamma)$, can be estimated as follows:

$$\begin{aligned} k := L(\Gamma) &\leq \|D\| \|A^{-1}\| + \epsilon \\ &\leq \mu\nu + \epsilon, \end{aligned}$$

where $\mu = \|Tf_{-1}|_{F^{uu}}\| < 1$ and $\nu = \|Tf_1|_{F^{ss}}\| < 1$. On the other hand, for the base map h we have the following estimate:

$$\ell := L(h^{-1}) = L(f_1) \leq \|Tf_1|_{F^{uu}}\|.$$

If ϵ is sufficiently small, then $k\ell < 1$. By the C^1 section theorem, it follows that Γ has a unique invariant section which is of class C^1. Since that section must be F^{ss}, the proof of the corollary is complete. □

In [**HK**], Hurder and Katok asked whether a similar effect can be achieved by reparametrizing an Anosov flow with the same properties as above but in dimension three. This was answered positively by Ghys in [**Gh2**]. Our corollary gives a positive answer to a similar question in dimensions > 3.

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