

HÖLDER FORMS AND INTEGRABILITY OF INVARIANT DISTRIBUTIONS

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ABSTRACT. We prove an inequality for Hölder continuous differential forms on compact manifolds in which the integral of the form over the boundary of a sufficiently small, smoothly immersed disk is bounded by a certain multiplicative convex combination of the volume of the disk and the area of its boundary. This inequality has natural applications in dynamical systems, where Hölder continuity is ubiquitous. We give two such applications. In the first one, we prove a criterion for the existence of global cross sections to Anosov flows in terms of their expansion-contraction rates. The second application provides an analogous criterion for non-accessibility of partially hyperbolic diffeomorphisms.

1. Introduction. Hölder continuity is ubiquitous in dynamical systems. Hölder continuous differential (though not differentiable) forms consequently play an important role there, especially in hyperbolic and partially hyperbolic dynamics. For instance, integrability of various invariant distributions (by which we mean bundles or plane fields) and the holonomy of the corresponding foliations can be expressed in terms of differential forms. Anosov used differential forms extensively for this purpose in his seminal paper [1].

Assume, for example, that $TM = E \oplus F$ is a Hölder continuous invariant splitting for a diffeomorphism $f : M \rightarrow M$. It is often important to know whether E is an integrable distribution. One can locally define $k = \dim F$ independent Hölder 1-forms $\alpha_1, \dots, \alpha_k$ such that the intersection of their kernels equals E . If F admits a global frame, these forms can be defined globally. Let $\alpha = \alpha_1 \wedge \dots \wedge \alpha_k$. Then $i_v \alpha = 0$, for every vector v tangent to E (where i_v denotes the inner multiplication by v), so we can write $\text{Ker}(\alpha) = E$. The Frobenius integrability condition (see, e.g., [21]) requires that $d\alpha$ be divisible by α , i.e., that $d\alpha = \alpha \wedge \omega$, for some 1-form ω . Recall that this is equivalent to $d\alpha_i \wedge \alpha = 0$, for all $1 \leq i \leq k$. Since α is only Hölder, this condition clearly does not apply. Hartman [10] (see also Plante [16]) proved an analogous integrability condition for continuous forms using the notion of the Stokes differential. Namely, α is said to be Stokes differentiable if there exists a locally integrable $(k + 1)$ -form β such that

$$\int_{\partial D} \alpha = \int_D \beta,$$

2000 *Mathematics Subject Classification.* Primary: 37D20, 37D30; Secondary: 49Q15.
Key words and phrases. Hölder forms, invariant distribution, global cross section.

for every smoothly immersed $(k + 1)$ -disk with piecewise smooth boundary ∂D . The form β is then called the Stokes differential of α . Hartman showed that E is integrable if and only if α divides β in the above sense. The utility of this result is limited since there are no good criteria for the Stokes differentiability of continuous Hölder forms.

In certain dynamical situations, however, in order to show integrability of an invariant distribution one needs less than the Stokes or Frobenius theorem, as we will demonstrate in this paper. The crucial inequality is given in Theorem A: for any compact manifold M , there exist numbers $K, \sigma > 0$ depending on M, k , and $\theta \in (0, 1)$, such that for every C^θ Hölder k -form ($1 \leq k \leq n-1$) α on M and every C^1 immersed disk D with piecewise C^1 boundary satisfying $\max\{\text{diam}(\partial D), |\partial D|\} < \sigma$,

$$\left| \int_{\partial D} \alpha \right| \leq K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta.$$

The idea behind the proof is simple: we approximate α locally by smooth forms α^ε , such that $\|\alpha - \alpha^\varepsilon\|_{C^0} \leq \|\alpha\|_{C^\theta} \varepsilon^\theta$ and $\|d\alpha^\varepsilon\|_{C^0} \leq C \|\alpha\|_{C^\theta} \varepsilon^{\theta-1}$, where $C > 0$ is a universal constant. This is done by using the standard technique of regularization. By subtracting and adding α^ε from and to α in $\int_{\partial D} \alpha$, it is easy to show that

$$\left| \int_{\partial D} \alpha \right| \leq C(\theta) \|\alpha\|_{C^\theta} (|\partial D| \varepsilon^\theta + |D| \varepsilon^{\theta-1}), \quad (1.1)$$

where $C(\theta)$ is a constant depending only on θ . The trick is to allow ε to vary over a sufficiently large interval and then find the best ε by minimizing the right-hand side of (1.1). For this, $|D|/|\partial D|$ needs to be sufficiently small. To eliminate the smallness requirement on $|D|/|\partial D|$, we use a special case of the isoperimetric inequality on Riemannian manifolds, supplied by Gromov [7].

We give two applications of this inequality in dynamical systems. First, we prove a criterion for the existence of a global cross section to an Anosov flow in terms of its expansion and contraction rates (Theorem B). We then translate this result into the language of partially hyperbolic diffeomorphisms and give a criterion for *non-accessibility*, also in terms of expansion-contraction rates (Theorem C). Both applications have strong limitations in that they apply only to a “small” set of systems. This is not surprising, since “most” distributions are not integrable. However, Theorem C suggests that there is a certain trade-off between the size of the Hölder exponent θ of the invariant splitting and accessibility: if θ is better than the standard lower estimate, accessibility is lost. This is illustrated by an example, due to an anonymous referee.

Notation. If S is a k -dimensional immersed submanifold of a Riemannian manifold M , $|S|$ will always denote its Riemannian k -volume. If $f : M \rightarrow N$ is a smooth map between smooth manifolds, $T_p f$ will denote its derivative (or tangent map) $T_p M \rightarrow T_{f(p)} N$. For non-negative functions u, v , we will write $u \lesssim v$ if there exists a uniform constant $c > 0$ such that $u \leq cv$. If $u \lesssim v$ and $v \lesssim u$, we write $u \asymp v$.

If (X, d) is a metric space and $0 < \theta < 1$, recall that a function $f : X \rightarrow \mathbb{R}$ is called C^θ Hölder (or just C^θ for short) if

$$H_\theta(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\theta} < \infty.$$

The C^θ -norm of f is defined by

$$\|f\|_{C^\theta} = \sup_{x \in X} |f(x)| + H_\theta(f).$$

Acknowledgments. We thank the anonymous referees for constructive criticism and the example at the end of the paper. While the paper was being revised for publication, Jenny Harrison pointed out that there are connections between Theorem A and some of her work in [9].

2. Preliminaries. We start with a short overview of regularization of functions and the isoperimetric inequality.

2-A. Regularization in \mathbb{R}^n . We briefly review a well-known method of approximating locally integrable functions by smooth ones, which will be used in the proof of the main inequality.

Suppose $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally integrable and define its regularization (or mollification) as the convolution

$$u^\varepsilon = \eta_\varepsilon * u,$$

where $\eta_\varepsilon(x) = \varepsilon^{-n} \eta\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, and $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ is the standard mollifier [5, 20]

$$\eta(x) = \begin{cases} A \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

with A chosen so that $\int \eta \, dx = 1$. Note that the support of η_ε is contained in the ball of radius ε centered at 0 and $\int \eta_\varepsilon \, dx = 1$.

Proposition 2.1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally integrable. Then:*

- (a) $u^\varepsilon \in C^\infty(\mathbb{R}^n)$.
- (b) If $u \in L^\infty$, then $\|u^\varepsilon\|_{L^\infty} \leq \|u\|_{L^\infty}$.
- (c) If $u \in C^\theta$ ($0 < \theta \leq 1$), then $\|u^\varepsilon - u\|_{C^0} \leq \|u\|_{C^\theta} \varepsilon^\theta$.
- (d) If $u \in C^\theta$, then

$$\|du^\varepsilon\|_{C^0} \leq \|d\eta\|_{L^1} \|u\|_{C^\theta} \varepsilon^{\theta-1}$$

$$\text{where } \|d\eta\|_{L^1} = \max_i \int_{\mathbb{R}^n} |\partial\eta/\partial x_i| \, dx.$$

Proof. Proof of (a) and (b) can be found in [5]. For (c), we have

$$\begin{aligned} |u^\varepsilon(x) - u(x)| &= \left| \int_{B(0,\varepsilon)} \eta_\varepsilon(y) [u(x-y) - u(x)] \, dy \right| \\ &\leq \|u\|_{C^\theta} \varepsilon^\theta \int_{B(0,\varepsilon)} \eta_\varepsilon(y) \, dy \\ &= \|u\|_{C^\theta} \varepsilon^\theta. \end{aligned}$$

If $u \in C^1$, then the same estimates hold with θ replaced by 1.

Observe that since η_ε has compact support,

$$\int_{\mathbb{R}^n} \frac{\partial \eta_\varepsilon}{\partial x_i}(y) \, dy = 0, \tag{2.1}$$

for $1 \leq i \leq n$. Note also that

$$\frac{\partial \eta_\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon^{n+1}} \frac{\partial \eta}{\partial x_i}\left(\frac{x}{\varepsilon}\right).$$

Assuming $u \in C^\theta$, we obtain (d):

$$\begin{aligned}
\left| \frac{\partial u^\varepsilon}{\partial x_i}(x) \right| &= \left| \int_{\mathbb{R}^n} u(x-y) \frac{\partial \eta_\varepsilon}{\partial x_i}(y) dy \right| \\
&\stackrel{\text{by (2.1)}}{=} \left| \int_{B(0,\varepsilon)} [u(x-y) - u(x)] \frac{\partial \eta_\varepsilon}{\partial x_i}(y) dy \right|, \\
&\leq \|u\|_{C^\theta} \varepsilon^\theta \int_{B(0,\varepsilon)} \left| \frac{\partial \eta_\varepsilon}{\partial x_i}(y) \right| dy \\
&= \|u\|_{C^\theta} \varepsilon^\theta \int_{B(0,\varepsilon)} \frac{1}{\varepsilon^{n+1}} \left| \frac{\partial \eta}{\partial x_i} \left(\frac{y}{\varepsilon} \right) \right| dy \\
&\stackrel{z=\frac{y}{\varepsilon}}{=} \|u\|_{C^\theta} \varepsilon^\theta \cdot \frac{1}{\varepsilon} \int_{B(0,1)} \left| \frac{\partial \eta}{\partial x_i}(z) \right| dz, \\
&\leq \|d\eta\|_{L^1} \|u\|_{C^\theta} \varepsilon^{\theta-1}. \quad \square
\end{aligned}$$

If α is a k -form on an open set $U \subset \mathbb{R}^n$, then α can be regularized component-wise. Write

$$\alpha = \sum_I a_I dx_I,$$

where $I = (i_1, \dots, i_k)$, $i_1 < \dots < i_k$, $i_1, \dots, i_k \in \{1, \dots, n\}$, and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. By definition, α is of class C^r ($r \in [0, \infty]$) if and only if each a_I is of class C^r . Define the ε -regularization of α by

$$\alpha^\varepsilon = \sum_I a_I^\varepsilon dx_I, \quad (2.2)$$

where for each I , a_I^ε is the ε -regularization of a_I .

2-B. Regularization on Riemannian manifolds. Suppose now that M is a compact C^∞ Riemannian manifold and fix a finite atlas $\mathcal{A} = \{(U, \varphi)\}$ of M . Let α be a C^θ k -form on M . This means that α is of class C^θ in every local coordinate system on M , i.e., $(\varphi^{-1})^* \alpha$ is C^θ on $\varphi(U)$, for each chart $(U, \varphi) \in \mathcal{A}$. Define the C^θ -norm of α by

$$\|\alpha\|_{C^\theta} = \max_{(U, \varphi) \in \mathcal{A}} \|(\varphi^{-1})^* \alpha\|_{C^\theta(\varphi(U))}. \quad (2.3)$$

We regularize α in each coordinate chart as follows. For each $(U, \varphi) \in \mathcal{A}$, choose an open set \hat{U} in M such that the closure of \hat{U} is contained in U and the collection $\{\hat{U}\}$ still covers M . Since M is compact, without loss of generality we can assume that $\varphi(U)$ – hence $\varphi(\hat{U})$ – is bounded. Define

$$\varepsilon_M = \min_{(U, \varphi) \in \mathcal{A}} \inf \{d(x, y) : x \in \partial\varphi(U), y \in \partial\varphi(\hat{U})\}. \quad (2.4)$$

If the representation of α in the (U, φ) -coordinates is

$$\tilde{\alpha}_U = (\varphi^{-1})^* \alpha = \sum_I a_I dx_I,$$

then we define the ε -regularization of α on \hat{U} by $\alpha_U^\varepsilon = \varphi^*(\tilde{\alpha}_U^\varepsilon)$, where $\tilde{\alpha}_U^\varepsilon$ is the ε -regularization of $\tilde{\alpha}_U$ defined as in (2.2). If $\varepsilon \in (0, \varepsilon_M)$, then α_U^ε is defined on \hat{U} , for every $(U, \varphi) \in \mathcal{A}$.

This produces a family $\{\alpha_U^\varepsilon\}$ of C^∞ k -forms, with α_U^ε approximating α on \hat{U} in the sense of Proposition 2.1. Using partitions of unity, this family can be patched

together into a globally defined smooth form; however, for our purposes local regularization will be sufficient.

2-C. Remarks on the isoperimetric inequality. We will also need a special case of the isoperimetric inequality on Riemannian manifolds, which we now briefly review.

Recall that the isoperimetric inequality for \mathbb{R}^n states (cf., [14, 8]) that for an arbitrary domain D in \mathbb{R}^n , its n -volume $|D|$ and the $(n-1)$ -volume $|\partial D|$ of its boundary are related as

$$|D| \leq C_n |\partial D|^{\frac{n}{n-1}}, \quad (2.5)$$

where

$$C_n = \frac{1}{(n^n \omega_n)^{1/(n-1)}},$$

and ω_n denotes the volume of the unit ball in \mathbb{R}^n .

On Riemannian manifolds the situation is more complicated, so we will only discuss a special case we need in this paper.

Given a cycle Z ($\partial Z = 0$) in a Riemannian manifold M , recall that the **isoperimetric problem** asks whether there is a volume minimizing chain Y in M , such that $\partial Y = Z$. For our purposes it suffices to consider this problem for *small* Z . The solution is given by the following result.

Lemma 2.2 ([7], **Sublemma 3.4.B'**). *For every compact manifold M , there exists a small positive constant δ_M such that every k -dimensional cycle Z in M of volume less than δ_M bounds a chain Y in M , which is small in the following sense:*

- (i) $|Y| \leq c_M |Z|^{(k+1)/k}$, for some constant c_M depending only on M ;
- (ii) The chain Y is contained in the ϱ -neighborhood of Z , where $\varrho \leq c_M |Z|^{1/k}$.

The following corollary is immediate.

Corollary 2.3. *If D is a C^1 -immersed $(k+1)$ -dimensional disk with piecewise C^1 boundary in a compact manifold M with $|\partial D| < \delta_M$, then there exists a $(k+1)$ -disk $\tilde{D} \subset M$ such that $\partial \tilde{D} = \partial D$,*

$$|\tilde{D}| \leq c_M |\partial D|^{\frac{k+1}{k}}, \quad (2.6)$$

and \tilde{D} is contained in the ϱ -neighborhood of ∂D , where $\varrho \leq c_M |\partial D|^{1/k}$.

3. The Main Inequality. We now have a necessary set-up for proving our main inequality.

Theorem A. *Let M be a compact manifold and let α be a C^θ k -form on M , for some $0 < \theta < 1$ and $1 \leq k \leq n-1$. There exist constants $\sigma, K > 0$, depending only on M , θ , and k , such that for every C^1 -immersed $(k+1)$ -disk D in M with piecewise C^1 boundary satisfying $\max\{\text{diam}(\partial D), |\partial D|\} < \sigma$, we have*

$$\left| \int_{\partial D} \alpha \right| \leq K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta.$$

Proof. The proof is divided into three steps. First, we show that the inequality holds for small, sufficiently flat disks in \mathbb{R}^n . By sufficiently flat, we mean that the ratio $|D|/|\partial D|$ is small enough. Second, we extend this result to compact manifolds.

Finally, we use the isoperimetric inequality to remove the smallness assumption on $|D|/|\partial D|$ and complete the proof of the theorem.

Step I. We now prove the inequality for small, sufficiently flat disks in \mathbb{R}^n . Let U, \hat{U} be bounded (open) domains in \mathbb{R}^n such that the closure of \hat{U} is contained in U . Define

$$\hat{\varepsilon} = \inf\{d(x, y) : x \in \partial U, y \in \partial \hat{U}\}.$$

Observe that $0 < \hat{\varepsilon} < \infty$. Let α be a C^θ k -form defined on U . Then we have:

Proposition 3.1. *For every C^1 -immersed $(k+1)$ -disk D in \hat{U} with piecewise C^1 boundary, satisfying*

$$\frac{|D|}{|\partial D|} < \frac{\theta \hat{\varepsilon}}{1 - \theta},$$

we have

$$\left| \int_{\partial D} \alpha \right| \leq C(\theta) \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta,$$

where

$$C(\theta) = \max\{1, \|d\eta\|_{L^1}\} \left\{ \left(\frac{1-\theta}{\theta} \right)^\theta + \left(\frac{\theta}{1-\theta} \right)^{1-\theta} \right\}.$$

Here η is as in §2-A.

Proof. Let α^ε be the ε -regularization of α as above. If $0 < \varepsilon < \hat{\varepsilon}$, then α^ε is defined on \hat{U} . Furthermore, by Proposition 2.1

$$\|\alpha - \alpha^\varepsilon\|_{C^0} \leq \|\alpha\|_{C^\theta} \varepsilon^\theta \quad \text{and} \quad \|d\alpha^\varepsilon\|_{C^0} \leq \|d\eta\|_{L^1} \|\alpha\|_{C^\theta} \varepsilon^{\theta-1}.$$

Let D be a $(k+1)$ -disk in \hat{U} satisfying $|D|/|\partial D| < \theta \hat{\varepsilon}/(1-\theta)$. Subtracting and adding α^ε , and using the Stokes theorem, we obtain:

$$\begin{aligned} \left| \int_{\partial D} \alpha \right| &\leq \left| \int_{\partial D} (\alpha - \alpha^\varepsilon) \right| + \left| \int_{\partial D} \alpha^\varepsilon \right| \\ &= \left| \int_{\partial D} (\alpha - \alpha^\varepsilon) \right| + \left| \int_D d\alpha^\varepsilon \right| \\ &\leq |\partial D| \|\alpha\|_{C^\theta} \varepsilon^\theta + |D| \|d\eta\|_{L^1} \|\alpha\|_{C^\theta} \varepsilon^{\theta-1} \\ &\leq \max\{1, \|d\eta\|_{L^1}\} \|\alpha\|_{C^\theta} (|\partial D| \varepsilon^\theta + |D| \varepsilon^{\theta-1}). \end{aligned}$$

The estimate is valid for all ε for which α^ε is defined on \hat{U} , that is, for $0 < \varepsilon < \hat{\varepsilon}$. The minimum of

$$\varepsilon \mapsto |\partial D| \varepsilon^\theta + |D| \varepsilon^{\theta-1}$$

is achieved at $\varepsilon_* = (1-\theta)|D|/(\theta|\partial D|)$, which lies in the permissible range $(0, \hat{\varepsilon})$. This minimum equals

$$\left\{ \left(\frac{1-\theta}{\theta} \right)^\theta + \left(\frac{\theta}{1-\theta} \right)^{1-\theta} \right\} |\partial D|^{1-\theta} |D|^\theta. \quad \square$$

Remark. If α is C^1 , then it is C^θ , for all $0 < \theta < 1$, and it is not hard to check that as $\theta \rightarrow 1-$,

$$C(\theta) \rightarrow \max\{1, \|d\eta\|_{L^1}\}.$$

Step II. Let M be a compact Riemannian C^∞ manifold. We fix an atlas $\mathcal{A} = \{(U, \varphi)\}$ such that each $\varphi(U)$ is bounded. For each chart $(U, \varphi) \in \mathcal{A}$, choose an open set $\hat{U} \subset U$ so that:

- the closure of \hat{U} is contained in U ;
- the collection $\{\hat{U}\}$ covers M .

Let ε_M be defined as in (2.4) and denote the Lebesgue number of the covering $\{\hat{U}\}$ by L . This means that for every set $S \subset M$, if $\text{diam}(S) < L$, then $S \subset \hat{U}$, for some chart U .

Define also

$$b_- = \min_{(U, \varphi) \in \mathcal{A}} \inf_{p \in \hat{U}} \|T_p \varphi\| \quad \text{and} \quad b_+ = \max_{(U, \varphi) \in \mathcal{A}} \sup_{p \in \hat{U}} \|T_p \varphi\|.$$

Since \mathcal{A} is finite and the sets \hat{U} are relatively compact, b_- and b_+ are finite and positive.

Proposition 3.2. *If α is a C^θ k -form on M , then for every C^1 -immersed $(k+1)$ -disk D with piecewise C^1 boundary in M satisfying $\text{diam}(D) < L$ and*

$$\frac{|D|}{|\partial D|} < \frac{b_-^k \theta \varepsilon_M}{(1-\theta) b_+^{k+1}},$$

we have

$$\left| \int_{\partial D} \alpha \right| \leq \varkappa C(\theta) \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta,$$

where $C(\theta)$ is the same as above, $\|\alpha\|_{C^\theta}$ was defined in (2.3), and \varkappa is a constant depending only on M , θ , and k .

Proof. Let D be a disk satisfying the above assumptions. Since $\text{diam}(D) < L$, there exists a chart U such that $D \subset \hat{U}$. Observe that

$$\frac{|\varphi(D)|}{|\partial \varphi(D)|} \leq \frac{b_+^{k+1} |D|}{b_-^k |\partial D|} < \frac{\theta \varepsilon_M}{1-\theta}.$$

Therefore, we can use the change of variables formula and apply Proposition 3.1 to $(\varphi^{-1})^* \alpha$ on $\varphi(D)$. We obtain:

$$\begin{aligned} \left| \int_{\partial D} \alpha \right| &= \left| \int_{\partial \varphi(D)} (\varphi^{-1})^* \alpha \right| \\ &\leq C(\theta) \|\alpha\|_{C^\theta} |\partial \varphi(D)|^{1-\theta} |\varphi(D)|^\theta \\ &\leq C(\theta) \|\alpha\|_{C^\theta} (b_+^k |\partial D|)^{1-\theta} (b_+^{k+1} |D|)^\theta \\ &= C(\theta) \|\alpha\|_{C^\theta} b_+^{k+\theta} |\partial D|^{1-\theta} |D|^\theta. \end{aligned}$$

This completes the proof of the proposition with $\varkappa = b_+^{k+\theta}$. \square

Step III. To extend the inequality to all small disks, we proceed as follows. Let

$$\sigma = \min \left\{ 1, \delta_M, \left(\frac{b_-^k \theta \varepsilon_M}{(1-\theta) b_+^{k+1} c_M} \right)^k, \left(\frac{L}{1+c_M} \right)^k \right\}.$$

Here δ_M and c_M are the same as in Lemma 2.2. Suppose that D satisfies $\max\{\text{diam}(\partial D), |\partial D|\} < \sigma$. Then by Corollary 2.3, there exists a $(k+1)$ -disk \tilde{D} such that $\partial\tilde{D} = \partial D$,

$$|\tilde{D}| \leq c_M |\partial\tilde{D}|^{(k+1)/k},$$

and \tilde{D} is contained in the ϱ -neighborhood of ∂D , with $\varrho \leq c_M |\partial D|^{1/k} < c_M \sigma^{1/k}$. If $|D| \leq |\tilde{D}|$, we can simply take $\tilde{D} = D$, so without loss we assume $|D| > |\tilde{D}|$.

The above assumptions imply

$$\frac{|\tilde{D}|}{|\partial\tilde{D}|} \leq c_M |\partial D|^{1/k} < \frac{b_-^k \theta \varepsilon_M}{(1-\theta)b_+^{k+1}},$$

and

$$\begin{aligned} \text{diam}(\tilde{D}) &\leq \text{diam}(\partial D) + \varrho \\ &\leq \sigma + c_M \sigma^{1/k} \\ &\leq \sigma^{1/k} (1 + c_M) \\ &< L \end{aligned}$$

so we can apply Proposition 3.2 to α on \tilde{D} . This yields

$$\begin{aligned} \left| \int_{\partial D} \alpha \right| &= \left| \int_{\partial\tilde{D}} \alpha \right| \\ &\leq \varkappa C(\theta) \|\alpha\|_{C^\theta} |\partial\tilde{D}|^{1-\theta} |\tilde{D}|^\theta \\ &= \varkappa C(\theta) \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |\tilde{D}|^\theta \\ &< \varkappa C(\theta) \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta. \quad \square \end{aligned}$$

Remark. (a) If $k = 1$, then $\text{diam}(\partial D) \leq |\partial D|$, so the assumption $\text{diam}(\partial D) < \sigma$ is superfluous.

(b) The estimate also holds for “long, thin” disks D , namely, those that can be decomposed into finitely many small disks D_1, \dots, D_N such that $|D_i| \lesssim |D|/N$, $|\partial D| \lesssim |\partial D|/N$ and Theorem A applies to each D_i . For then $\partial D = \partial D_1 + \dots + \partial D_N$ and

$$\begin{aligned} \left| \int_{\partial D} \alpha \right| &= \left| \sum_{i=1}^N \int_{\partial D_i} \alpha \right| \\ &\leq \sum_{i=1}^N K \|\alpha\|_{C^\theta} |\partial D_i|^{1-\theta} |D_i|^\theta \\ &\lesssim NK \|\alpha\|_{C^\theta} \left(\frac{|\partial D|}{N} \right)^{1-\theta} \left(\frac{|D|}{N} \right)^\theta \\ &= K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta. \end{aligned}$$

(c) Theorem A is also valid for immersed *submanifolds* D with piecewise smooth boundary. The proof goes through word for word.

4. Global cross sections to Anosov flows. Recall that a non-singular smooth flow $\Phi = \{f_t\}$ on a closed (compact and without boundary) Riemannian manifold M is called Anosov if there exists a Tf_t -invariant continuous splitting of the tangent bundle,

$$TM = E^{ss} \oplus E^c \oplus E^{uu},$$

and constants $C > 0$, $0 < \nu < 1$, and $\lambda > 1$ such that for all $t \geq 0$,

$$\|Tf_t \upharpoonright_{E^{ss}}\| \leq C\nu^t \quad \text{and} \quad \|Tf_t \upharpoonright_{E^{uu}}\| \geq C\lambda^t.$$

The center bundle E^c is one dimensional and generated by the vector field X tangent to the flow. The distributions E^{uu} , E^{ss} , $E^{cu} = E^c \oplus E^{uu}$, and $E^{cs} = E^c \oplus E^{ss}$ are called the strong unstable, strong stable, center unstable, and center stable bundles, respectively. Typically they are only Hölder continuous [12, 11], yet uniquely integrable [1], giving rise to continuous foliations denoted by W^{uu} , W^{ss} , W^{cu} , and W^{cs} , respectively. Recall that a distribution E is called **uniquely integrable** if it is tangent to a foliation and every differentiable curve everywhere tangent to E is wholly contained in a leaf of the foliation.

The idea of studying the dynamics of a flow by introducing a (local or global) cross section dates back to Poincaré. Recall that a smooth compact codimension one submanifold Σ of M is called a **global cross section** for a flow if it intersects every orbit transversely. If this is the case, then every point $p \in \Sigma$ returns to Σ , defining the Poincaré or first-return map $g : \Sigma \rightarrow \Sigma$. The flow can then be reconstructed by suspending g under the roof function equal to the first-return time [6, 13, 19].

The Poincaré map of a global cross section to an Anosov flow is automatically an Anosov diffeomorphism. Therefore, any classification of Anosov diffeomorphisms immediately translates into a classification of the corresponding class of suspension Anosov flows.

Geometric criteria for the existence of global cross sections to Anosov flows were obtained by Plante [16], who showed that the flow admits a smooth global cross section if the distribution $E^{su} = E^{ss} \oplus E^{uu}$ is (uniquely) integrable. He also showed that E^{su} is integrable if and only if the foliations W^{ss} and W^{uu} are jointly integrable. This means that in every joint foliation chart for W^{cs} and W^{uu} , the W^{uu} -holonomy takes W^{ss} -plaques to W^{ss} -plaques. Joint integrability of W^{uu} and W^{ss} (in that order) is defined analogously; by symmetry, E^{su} is uniquely integrable if and only if W^{uu} and W^{ss} are jointly integrable.

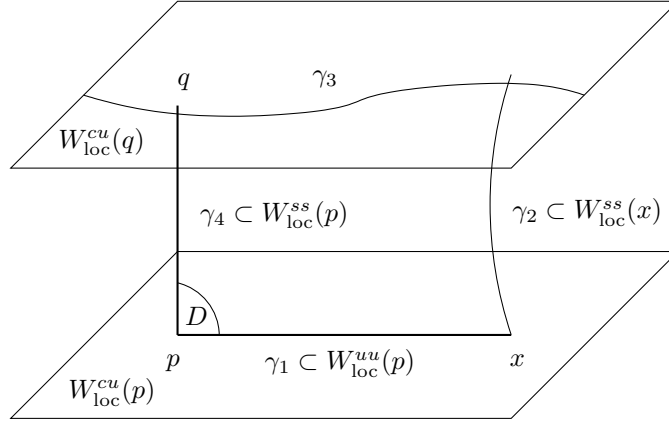
We now present a criterion for the existence of a global cross section to an Anosov flow in terms of its expansion-contraction rates.

Let p and q be in the same local strong unstable manifold of an Anosov flow $\Phi = \{f_t\}$. Assume p and q are close enough so that they lie in a foliation chart for both W^{cu} and W^{ss} .

Definition 4.1. *If E^{ss} is of class C^1 , a C^1 immersed 2-disk $D \subset M$ is called a **us-disk** if:*

- *Its boundary is the concatenation of four simple C^1 paths: $\partial D = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where $\gamma_1 \subset W_{\text{loc}}^{uu}(p)$, $\gamma_3 \subset W_{\text{loc}}^{cu}(q)$, for some p, q as above; furthermore, $\gamma_2 \subset W_{\text{loc}}^{ss}(x)$ and $\gamma_4 \subset W_{\text{loc}}^{ss}(p)$, where x is the terminal point of γ_1 .*
- *D is a union of W^{ss} -arcs, i.e., arcs contained in the strong stable plaques.*

*We will call γ_1 the **base** of D and $\gamma_2, \gamma_3, \gamma_4$ its **sides**. See Fig. 1.*

FIGURE 1. A us -disk D with base γ_1 .

Define a 1-form α on M by requiring

$$\text{Ker}(\alpha) = E^{ss} \oplus E^{uu}, \quad \alpha(X) = 1. \quad (4.1)$$

Since $E^{ss} \oplus E^{uu}$ is of class C^θ , so is α . It is clear that α is invariant under the flow: $f_t^* \alpha = \alpha$, for all $t \in \mathbb{R}$.

Lemma 4.2. *If E^{ss} is of class C^1 , then the following statements are equivalent.*

- (a) W^{uu} and W^{ss} are jointly integrable.
- (b) $\int_{\partial D} \alpha = 0$, for every us -disk D .

Proof. Follows directly from the definitions of α and joint integrability of W^{uu} and W^{ss} . \square

Theorem B. *Suppose $\Phi = \{f_t\}$ is a C^2 Anosov flow on a closed Riemannian manifold M . Assume:*

- (a) $\|Tf_t|_{E^{uu}}\| \leq C\mu^t$ and $\|Tf_t|_{E^{ss}}\| \leq C\nu^t$, for all $t \geq 0$, and some constants $C > 0$, $\mu > 1$, and $0 < \nu < 1$.
- (b) E^{ss} is of class C^1 .
- (c) $\mu\nu^\theta < 1$, where $\theta \in (0, 1)$ is the Hölder exponent of E^{uu} .

Then Φ admits a global cross section.

Proof. Let D be an us -disk with base $\gamma_1 : [0, 1] \rightarrow W_{\text{loc}}^{uu}(p)$ and sides γ_i , $i = 2, 3, 4$ as above. Then:

Lemma 4.3. $|f_t D| \lesssim (\mu\nu)^t |D|$, for all $t \geq 0$.

Proof. There exists a C^1 vector field Y tangent to E^{ss} with flow $\{\psi_s\}$ such that D can be parametrized by

$$\Psi(r, s) = \psi_s(\gamma_1(r)), \quad 0 \leq r \leq 1, \quad 0 \leq s \leq \tau(r),$$

for some continuous function $\tau : [0, 1] \rightarrow \mathbb{R}$. Since $f_t \circ \Psi$ is a parametrization of $f_t D$, the area element of $f_t D$ is

$$\left\| Tf_t \left(\frac{\partial \Psi}{\partial r} \wedge \frac{\partial \Psi}{\partial s} \right) \right\|.$$

By the chain rule,

$$\frac{\partial \Psi}{\partial r} = T\psi_s(\dot{\gamma}_1(r)), \quad \frac{\partial \Psi}{\partial s} = Y.$$

The vector $T\psi_s(\dot{\gamma}_1(r))$ decomposes into $w_{ss} + w_c + w_{uu}$ relative to the splitting $E^{ss} \oplus E^c \oplus E^{uu}$. Since $\|Tf_t(w_{ss})\| \leq C\nu^t \|w_{ss}\| \rightarrow 0$, as $t \rightarrow \infty$, and $\|Tf_t(w_c)\|$ is constant, it follows that for $t \geq 0$,

$$\|Tf_t(T\psi_s(\dot{\gamma}_1(r)))\| \lesssim \mu^t.$$

Clearly, $\|Tf_t(\frac{\partial \Psi}{\partial s})\| = \|Tf_t(Y)\| \leq C\nu^t$. Therefore,

$$\left\| Tf_t \left(\frac{\partial \Psi}{\partial r} \wedge \frac{\partial \Psi}{\partial s} \right) \right\| = \|Tf_t(w_{ss} \wedge Y + w_c \wedge Y + w_{uu} \wedge Y)\|,$$

which is dominated by $\|Tf_t(w_{uu} \wedge Y)\| \lesssim \mu^t \nu^t$, as $t \rightarrow \infty$. This clearly implies the claim of the lemma. \square

By Lemma 4.2, we need to show that

$$\int_{\partial D} \alpha = 0,$$

for every us -disk D . It is enough to prove this for small D . The idea is to use the flow invariance of α and change of variables,

$$\int_{\partial D} \alpha = \int_{\partial f_t D} \alpha,$$

and then apply Theorem A to show that the right-hand side converges to zero, as $t \rightarrow \infty$. Since $f_t D$ is very “long”, we cannot use Theorem A directly. However, $f_t D$ is also very “thin”, since the length of its W^{ss} -sides go to zero, as $t \rightarrow \infty$. More precisely,

$$|\partial^u f_t D| \leq C\mu^t |\partial^u D|, \quad |\partial^s f_t D| \leq C\nu^t |\partial^s D|,$$

where $\partial^u D = \gamma_1 + \gamma_3$ and $\partial^s D = \gamma_2 + \gamma_4$. We therefore proceed by cutting D into N us -disks D_i , as in Figure 2. We decompose $\gamma := \gamma_1$ as $\gamma = \gamma_1 + \dots + \gamma_N$, so that for each $i = 1, \dots, N$, γ_i gives rise to a us -disk D_i with $|D_i| = |D|/N$. To determine how large N has to be as a function of $t > 0$, recall that we need $|\partial f_t D_i| < \sigma$, for each i , in order to apply Theorem A. Using the notation ∂^u, ∂^s (with clear meaning), for each i we have:

$$\begin{aligned} |\partial f_t D_i| &= |\partial^u f_t D_i| + |\partial^s f_t D_i| \\ &\leq C\mu^t |\partial^u D_i| + C\nu^t |\partial^s D_i| \\ &\leq CC_0 \frac{|\partial^u D|}{N} + C\nu^t |\partial^s D| \\ &\leq C_1 |\partial D| \left(\frac{\mu^t}{N} + \nu^t \right), \end{aligned}$$

where $C_0 > 1$ is some constant depending only on the hyperbolicity of the flow and $C_1 = CC_0$. So to ensure $|\partial f_t D_i| < \sigma$, we can take N to be of the order μ^t . More precisely, assuming t is so large that $C_1 |\partial D| \nu^t < \sigma/2$, choose N so that $C_1 |\partial D| \mu^t / N < \sigma/2$, i.e.,

$$N > \frac{2C_1 |\partial D| \mu^t}{\sigma} = N_0(t).$$

We also require that $N < 2N_0(t)$ so that $N \asymp \mu^t$.

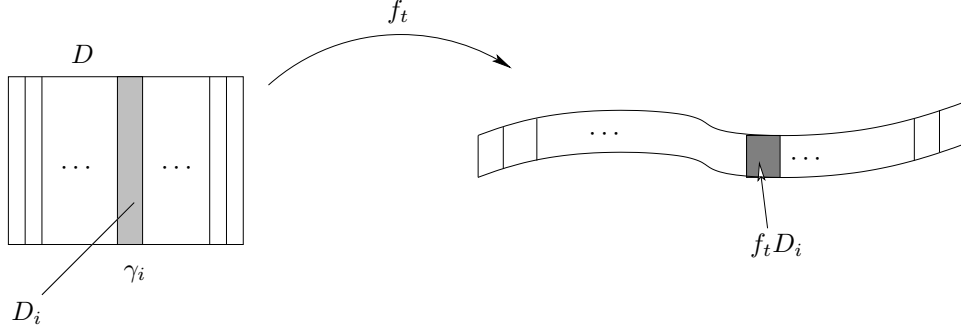


FIGURE 2. Decomposition of D into D_i 's.

By Lemma 4.3, we have

$$\begin{aligned}
 |f_t D_i| &\lesssim (\mu\nu)^t |D_i| \\
 &= (\mu\nu)^t \frac{|D|}{N} \\
 &\lesssim (\mu\nu)^t \frac{|D|}{\mu^t} \\
 &= \nu^t |D|.
 \end{aligned}$$

Applying Theorem A to each $f_t D_i$, we obtain the following estimate:

$$\begin{aligned}
 \left| \int_{\partial D} \alpha \right| &= \left| \int_{\partial f_t D} \alpha \right| \\
 &\leq \sum_{i=1}^N \left| \int_{\partial f_t D_i} \alpha \right| \\
 &\leq \sum_{i=1}^N K \|\alpha\|_{C^\theta} |\partial f_t D_i|^{1-\theta} |f_t D_i|^\theta \\
 &\lesssim N \cdot K \|\alpha\|_{C^\theta} \sigma^{1-\theta} (\nu^t |D|)^\theta \\
 &\lesssim K \|\alpha\|_{C^\theta} \sigma^{1-\theta} \mu^t \nu^{\theta t} |D|^\theta,
 \end{aligned}$$

since $N \lesssim \mu^t$. Letting $t \rightarrow \infty$, we obtain $\int_{\partial D} \alpha = 0$, as desired. \square

Remark. (a) It is likely that Theorem B could be slightly improved by using the extra smoothness of α along the leaves of the center unstable foliation. This extra smoothness comes from the fact that along the leaves of the center unstable foliation W^{cu} , the strong unstable distribution E^{uu} (assumed to be only Hölder) is actually as smooth as the flow, i.e., C^2 . The condition $\mu\nu^\theta < 1$ in Theorem B would then be replaced by a weaker one $\mu\nu^\tau < 1$, where $\tau = (2 - \theta)^{-1}$.

(b) It needs to be pointed out that the assumptions (b) and (c) of Theorem B are quite restrictive and are satisfied only by a small set of Anosov flows. However, once a system Φ does verify (b) and (c), by structural stability there exists

a C^1 neighborhood \mathcal{U} of Φ such that each flow in \mathcal{U} admits a global cross section.

5. Accessibility. In this section we prove a sufficient condition for a partially hyperbolic diffeomorphism to be non-accessible.

Recall that a diffeomorphism f of a compact Riemannian manifold M is called partially hyperbolic if the tangent bundle of M splits continuously and invariantly into the stable, center, and unstable bundle, $TM = E^s \oplus E^c \oplus E^u$, such that Tf exponentially contracts E^s , exponentially expands E^u and this hyperbolic action on $E^s \oplus E^u$ dominates the action of Tf on E^c . The stable and unstable bundles are always uniquely integrable, giving rise to the stable and unstable foliations, W^s, W^u . In contrast, the center bundle E^c , the center stable $E^{cs} = E^c \oplus E^s$, and the center unstable bundle $E^{cu} = E^c \oplus E^u$ are not always integrable. If they are, f is called dynamically coherent (cf., [3, 17, 4]).

A partially hyperbolic diffeomorphism is called accessible if every two points of M can be joined by an su -path, that is, a continuous piecewise smooth path consisting of finitely many arcs lying in a single leaf of W^s or a single leaf of W^u .

If f is dynamically coherent and the foliations W^s and W^u are jointly integrable (in the same sense as in Section §4), then it is clear that f is not accessible. We can also speak of joint integrability of W^u and W^s (in that order), which is defined analogously; it also implies non-accessibility.

Let $f : M \rightarrow M$ be a partially hyperbolic diffeomorphism with $\dim E^c = \ell$ and integrable center-unstable bundle E^{cu} . Assume that both E^s and E^u have dimension $\geq \ell$. We now define objects that will play the role of us -disks in this context.

Assume E^s is C^1 and pick an arbitrary $p \in M$ and $q \in W_{\text{loc}}^s(p)$. Let Γ be an ℓ -dimensional C^1 -immersed surface (with piecewise C^1 boundary) contained in $W_{\text{loc}}^u(p)$ and define D to be a C^1 -immersed $(\ell + 1)$ -disk (or ‘‘cube’’) by making the following requirements:

- $\partial D \cap W_{\text{loc}}^{cu}(p) = \Gamma$;
- D is foliated by arcs tangent to E^s ;
- $\partial D \cap W_{\text{loc}}^{cu}(q) = h^s(\Gamma)$, where $h^s : W_{\text{loc}}^{cu}(p) \rightarrow W_{\text{loc}}^{cu}(q)$ is the holonomy map associated with the stable foliation W^s .

We will call any disk (or ‘‘cube’’) D satisfying these requirements a us -cube for f . We will refer to Γ as the base of D . We also write $\partial^u D = \Gamma + h^s(\Gamma)$ and $\partial^s D = \partial D - \partial^u D$.

Denote the restriction of Tf to E^ρ by $T^\rho f$, with $\rho \in \{s, c, u\}$, and by

$$m(T_p^c f) = \min\{\|T_p^c f(v)\| : v \in E^c(p), \|v\| = 1\},$$

the conorm (or minimum norm) of $T_p^c f$. We also define

$$\lambda_\rho = \|T^\rho f\| = \max\{\|T_p^\rho f\| : p \in M\},$$

where $\rho \in \{s, u\}$, and $m(T^c f) = \min\{m(T_p^c f) : p \in M\}$.

Theorem C. *Suppose that $f : M \rightarrow M$ is a C^2 partially hyperbolic diffeomorphism of a compact Riemannian manifold M . Assume:*

- (a) E^s is C^1 ;

Using $|\det T^c f^{-k}| \leq m(T^c f)^{-k\ell}$, we obtain:

$$\begin{aligned}
\left| \int_{\partial D} \alpha \right| &= \left| \int_{\partial f^k D} (f^{-k})^* \alpha \right| \\
&\leq \sum_{i=1}^N \left| \int_{\partial f^k D_i} (\det T^c f^{-k}) \alpha \right| \\
&\leq \sum_{i=1}^N K \|(\det T^c f^{-k}) \alpha\|_{C^\theta} |\partial f^k D_i|^{1-\theta} |f^k D_i|^\theta \\
&\lesssim N \cdot K \|\alpha\|_{C^\theta} m(T^c f)^{-k\ell} \sigma^{1-\theta} (\lambda_s^k |D|)^\theta \\
&\lesssim \lambda_u^{k\ell} \cdot K \|\alpha\|_{C^\theta} m(T^c f)^{-k\ell} \sigma^{1-\theta} \lambda_s^{\theta k} |D|^\theta \\
&= K \|\alpha\|_{C^\theta} \sigma^{1-\theta} \left(\frac{\lambda_u^\ell \lambda_s^\theta}{m(T^c f)^\ell} \right)^k |D|^\theta,
\end{aligned}$$

which converges to zero, as $k \rightarrow \infty$. This completes the proof. \square

Remark. (a) The assumption that E^c is trivial is only used to obtain a globally defined form α .

(b) We point out that condition (e) requires the Hölder exponent θ of E^u to be better than its expected value given by (see [18])

$$\frac{m(T^u f)}{\|T^c f\|} m(T^s f)^\theta > 1.$$

While this is a clear limitation of Theorem C, it also seems to suggest that there is a certain trade-off between accessibility and the size of θ . Namely, if θ exceeds its expected value, then accessibility is lost, as illustrated by the following example. We are grateful to an anonymous referee for suggesting it.

Example. Let ξ be the real root of the equation $p(x) = x^3 - x - 1 = 0$. Then $\xi > 1$ and the other two roots of p are complex conjugate numbers that lie inside the unit circle (in other words, ξ is a Pisot number). Denote by $\eta < 1$ their common absolute value and let A be the companion matrix for the polynomial p . Since the constant term of p is -1 , we have $\xi\eta^2 = |\det A| = 1$. The matrix A defines a linear Anosov diffeomorphism f_A of the torus \mathbb{T}^3 . Let $f_0 : \mathbb{T}^4 \rightarrow \mathbb{T}^4$ be the direct product of f_A^{-1} and the identity on S^1 . Then f_0 is a partially hyperbolic diffeomorphism, with one dimensional center and stable bundles, and the unstable bundle of dimension two. Following Brin [2, 15], one can make an arbitrarily small perturbation of f_0 in the C^∞ topology to obtain an *accessible* partially hyperbolic diffeomorphism f . In particular, f does not satisfy the conclusion of Theorem C. The only hypothesis f violates is (e), which means that

$$\frac{\|T^u f\|^\ell \|T^s f\|^\theta}{m(T^c f)^\ell} \geq 1,$$

where $\ell = \dim E^c = 1$ and $\theta \in (0, 1)$ is the Hölder exponent of E^u and E^c . To as close an approximation as one wishes, we have $\|T^s f\| = \xi^{-1}$, $\|T^u f\| = \eta^{-1}$, and $m(T^c f) = 1$. Therefore, up to a small error we obtain $\eta^{-1}\xi^{-\theta} \geq 1$, or $\eta\xi^\theta \leq 1$. On the other hand, the standard estimate of the Hölder exponent E^u gives a lower bound on θ : namely (again up to a small error), $1/(\xi^{-\theta}\eta^{-1}) < 1$, that is, $\eta\xi^\theta > 1$.

This suggests the following conclusion: if f_0 is perturbed so as to become accessible, then the Hölder exponent of the splitting has to become as small as the standard lower estimate allows; any Hölder continuity better than the worst case forces joint integrability of W^s and W^u .

REFERENCES

- [1] Dimitri V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proc. Steklov Math. Inst. **90** (1967), AMS Translations (1969).
- [2] Michael Brin, *The topology of group extensions of C systems*, Mat. Zametki **19** (1975), 453–465.
- [3] ———, *On dynamical coherence*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 395–401.
- [4] Keith Burns and Amie Wilkinson, *Dynamical coherence, accessibility and center bunching*, Discrete Contin. Dyn. Syst. **22** (2008), no. 1-2, 89–100.
- [5] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, AMS, 1998.
- [6] David Fried, *The geometry of cross sections to flows*, Topology **21** (1982), 353–371.
- [7] Mikhail Gromov, *Filling Riemannian manifolds*, J. Diff. Geometry **18** (1983), 1–147.
- [8] ———, *Metric structures for Riemannian and non-Riemannian spaces*, Modern Birkhäuser Classics, Birkhäuser, 2007.
- [9] Jenny Harrison, *Continuity of the integral as a function of the domain*, Journal of Geometric Analysis **8** (1998), no. 5, 769–795.
- [10] Philip Hartman, *Ordinary differential equations*, second ed., Classics in Applied Mathematics, vol. 38, SIAM, 2002.
- [11] Boris Hasselblatt, *Regularity of the Anosov splitting II*, Ergodic Theory Dynam. Systems (1997), 169–172.
- [12] Morris W. Hirsch, Charles C. Pugh, and Michael Shub, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin-New York, 1977.
- [13] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, 1995.
- [14] Robert Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1182–1238.
- [15] Yakov B. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2004.
- [16] Joseph Plante, *Anosov flows*, Amer. J. of Math. **94** (1972), 729–754.
- [17] Charles C. Pugh and Michael Shub, *Stable ergodicity*, Bull. Amer. Math. Soc. **41** (2004), 1–41.
- [18] Charles C. Pugh, Michael Shub, and Amie Wilkinson, *Hölder foliations*, Duke Math. J. **86** (1997), no. 3, 517–546.
- [19] Saul Schwartzman, *Asymptotic cycles*, Annals of Mathematics **66** (1957), 270–284.
- [20] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
- [21] Frank W. Warner, *Foundations of differentiable manifolds and lie groups*, Graduate Texts in Math., no. 94, Springer-Verlag, 1983.

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