

Lyapunov functions and Anosov flows

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Abstract

We show that if the codimension one Anosov flow Φ on a compact n -manifold M satisfies the so called condition (L), then there is a continuous Lyapunov function $g : \mathbf{R}^n \rightarrow \mathbf{R}$, where \mathbf{R}^n is the universal covering space of M , such that g strictly increases along the orbits of the lift of Φ and is constant on the leaves of the lift of the strong stable foliation of the “synchronization” (i.e. suitable reparametrization) of Φ . We also give a sufficient condition in terms of g for Φ to admit a global cross section.

1 Introduction and background

Our main concern in this paper is existence of global cross sections to codimension one Anosov flows. However, instead of attacking the problem directly (which we did elsewhere: see [S1], [S2]), we focus our attention to the lift of the codimension one Anosov flow to the universal covering space of the underlying manifold. But first, let us briefly recall the main notions.

A nonsingular C^1 flow $\{f_t\}$ on a compact connected Riemannian manifold M is called **Anosov** if the tangent bundle of M splits continuously into the Tf_t -invariant Whitney sum $TM = E^c \oplus E^{ss} \oplus E^{uu}$ such that the **strong stable bundle** E^{ss} is exponentially contracted and the **strong unstable bundle** E^{uu} is exponentially expanded by Tf_t for $t > 0$; $E^c = \mathbf{R}X$ is the line bundle tangent to the flow, where X is the generating vector field. We call $E^{cs} := E^c \oplus E^{ss}$ and $E^{cu} := E^c \oplus E^{uu}$ the **center stable** and **center unstable bundle**, respectively. It was shown by Anosov [An] that all the bundles mentioned above are uniquely integrable, that is tangent to unique foliations denoted here by W^{ss} , W^{uu} , W^{cs} and W^{cu} , respectively. For basic facts about Anosov flows see also [P11]. An Anosov flow is of **codimension one** if $\dim E^{uu} = 1$ or $\dim E^{ss} = 1$; we will always assume the former. It is well known [Ve] that if $\dim M > 3$, then any codimension one Anosov flow on M is topologically transitive; moreover, its center stable bundle E^{cs} is of class C^1 (also if $\dim M = 3$; see [HP] and also [HK]).

Verjovsky conjectured that if $\{f_t\}$ is Anosov of codimension one and $\dim M > 3$, then $\{f_t\}$ admits a global cross section and is therefore topologically conjugate to the suspension of a linear Anosov diffeomorphism on a torus. This conjecture was verified in several special cases; see [P12], [S1] and [S2]. In this paper we consider the universal covering $p : \tilde{M} \rightarrow M$ of

M and the lifts $\tilde{f}_t, \tilde{X}, \tilde{E}^{cs}, \tilde{W}^{cs}$, etc. by p of the corresponding objects on M . It was shown by Palmeira [Pa] (as a consequence of a more general result) that $\tilde{M} \approx \mathbf{R}^n$, and from now on we identify \tilde{M} with \mathbf{R}^n via this diffeomorphism.

Later on we will need the following notion of the generalized exterior differential of a differential form.

Definition A locally integrable form β of degree $r + 1$ is the **generalized differential** of a locally integrable r -form α if for every C^∞ form ϕ of degree $n - r - 1$ with compact support the following holds:

$$\int_{\mathbf{R}^n} \phi \wedge \beta = (-1)^{n-r} \int_{\mathbf{R}^n} d\phi \wedge \alpha.$$

It is easy to see by Stokes' theorem that if α is of class C^1 , then its ordinary differential is also its generalized differential. For details on this matter we refer the reader to [Rs], II.4.

2 Results

For the rest of the paper $\{f_t\}$ will denote a codimension one Anosov flow on a compact manifold M . Consider the following:

Condition (L) *The lift \tilde{W}^{cs} of the center stable foliation of $\{f_t\}$ is a trivial foliation, i.e. given by a C^1 submersion $\mathbf{R}^n \rightarrow \mathbf{R}$.*

This condition was studied in [Ve], and an incorrect proof of it was given. (More precisely, Proposition 3.4 of [Ve] is incorrect. This was pointed out to me by S. Fenley.) We do not know how to prove condition (L), but since it will lead to some interesting consequences, we will adopt it as an additional assumption. It is easy to see that (L) is satisfied if, for instance, \tilde{W}^{cs} admits a transversal intersecting all of its leaves. By Solodov [So] this is the case when the center of the fundamental group of M is free cyclic.

From now on we take (L) as our standing hypothesis. Thus without loss of generality we may assume that the covering projection p is of class C^1 and \tilde{W}^{cs} is the hyperplane foliation given by $x_n = \text{constant}$. Define a 1-form ω on M by requiring

$$\text{Ker}(\omega) = E^{cs}, \quad \omega(Y) = 1,$$

where Y is a continuous vector field in E^{uu} . (We made such a choice for the value of ω in the transverse direction to E^{cs} because we want formula (1) below to hold.) Since E^{cs} is a C^1 bundle, by being a little careful in choosing Y (for instance, take Y of the form $Z + V$, where $V \in E^{cs}$, and Z is a nonsingular C^∞ vector field transverse to E^{cs}), we can make sure that ω is of class C^1 . By Frobenius theorem for the integrable bundle E^{cs} , there exists a continuous 1-form η on M such that $d\omega = \omega \wedge \eta$. Define $u : M \rightarrow \mathbf{R}$ by $u = \eta(X)$. Also let $\lambda(x, t) = \det T_x f_t|_{E^{uu}}$, with respect to some previously chosen Riemann structure \mathcal{R} on M . Note that the form η is not unique: if $\eta' = \eta + h\omega$, then $\eta' \wedge \omega = \eta \wedge \omega$. However, thanks to the

fact that $\omega(X) = 0$, u does not depend on the choice of η . It was shown in [S2] that for all $x \in M$ and $t \in \mathbf{R}$,

$$\lambda(x, t) = \exp \left\{ \int_0^t u(f_s x) ds \right\}, \quad (1)$$

and it is possible to choose ω and a continuous Riemann structure \mathcal{R} on M such that the corresponding u is C^1 and strictly positive. Note that this implies that the flow is “immediately expanding” on E^{uu} .

Since the 1-forms $\tilde{\omega}$ and dx_n have the same nullspace, namely \tilde{E}^{cs} , there exists a continuous function $g : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\tilde{\omega} = g dx_n.$$

Without loss we will assume that g is everywhere positive; otherwise, replace ω by $-\omega$.

Now we can state the following result.

Theorem 1 (a) g has partial derivatives with respect to x_1, \dots, x_{n-1} and they are continuous.
(b) There exists a continuous function $b : \mathbf{R}^n \rightarrow \mathbf{R}$ such that

$$\tilde{\eta} = d^{cs}(\log g) + b(x) dx_n.$$

Here d^{cs} denotes the leafwise \tilde{W}^{cs} -differential: $d^{cs}(\log g) = \sum_{i=1}^{n-1} (\partial \log g / \partial x_i) dx_i$.

(c) We have: $u(p(x)) = \tilde{X}_x(\log g)$ and

$$\lambda(p(x), t) = \frac{g(\tilde{f}_t(x))}{g(x)},$$

for every $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$.

(d) For every deck transformation T and every $x \in \mathbf{R}^n$,

$$g(Tx) T'_n(x_n) = g(x).$$

(Here T_n denotes the n^{th} coordinate function of T . Since T preserves \tilde{W}^{cs} , T_n depends only on x_n , so $T'_n(x_n) = \frac{\partial T_n}{\partial x_n}(x_n)$.)

PROOF (a), (b) We will use the notion of generalized differentials defined above.

Consider the form $\tilde{\omega} = p^*\omega$. Since p is only of class C^1 , we cannot claim that $\tilde{\omega}$ is a C^1 form. However, using p as a local change of variables, it is not difficult to see that $\tilde{\omega}$ is actually differentiable in the generalized sense. Indeed, let ϕ be a C^∞ form of degree $n - 2$ with compact support in \mathbf{R}^n . Without loss of generality we can assume the support of ϕ is contained in a small open set U on which p is 1-1 (otherwise use a partition of unity). Thus there exists an $(n - 2)$ -form ψ on M such that $\phi = (p|_U)^*\psi$ and $d\psi$ exists in the generalized sense. It follows

$$\int_{\mathbf{R}^n} d\phi \wedge \tilde{\omega} = \int_U p^*(d\psi \wedge \omega)$$

$$\begin{aligned}
&= \int_M d\psi \wedge \omega \\
&= (-1)^{n-1} \int_M \psi \wedge d\omega \\
&= (-1)^{n-1} \int_{\mathbf{R}^n} \phi \wedge p^*(d\omega).
\end{aligned}$$

Therefore, $p^*(d\omega)$ is the generalized differential of $\tilde{\omega}$. Clearly, $p^*(d\omega) = p^*(\eta \wedge \omega) = \tilde{\eta} \wedge \tilde{\omega}$.

There exist continuous functions $a_i : \mathbf{R}^n \rightarrow \mathbf{R}$ ($1 \leq i \leq n$) such that

$$\tilde{\eta} = \sum_{i=1}^n a_i dx_i.$$

Let $h : \mathbf{R}^n \rightarrow \mathbf{R}$ be an arbitrary smooth function with compact support. Define $\phi = h dx_2 \wedge \dots \wedge dx_{n-1}$. Then we have:

$$\begin{aligned}
\int_{\mathbf{R}^n} g \frac{\partial h}{\partial x_1} dx_1 \wedge \dots \wedge dx_n &= \int_{\mathbf{R}^n} d\phi \wedge \tilde{\omega} \\
&= (-1)^{n-1} \int_{\mathbf{R}^n} \phi \wedge \tilde{\eta} \wedge \tilde{\omega} \\
&= - \int_{\mathbf{R}^n} g a_1 h dx_1 \wedge \dots \wedge dx_n.
\end{aligned}$$

This shows that the generalized (i.e. Sobolev) derivative of g with respect to x_1 is $g a_1$. Thus

$$a_1 = \frac{\partial(\log g)}{\partial x_1},$$

in the generalized sense. Similarly, we can prove analogous formulas for a_2, \dots, a_{n-1} . Since a_i 's are continuous hence locally bounded functions, g actually belongs to the Sobolev space $W_{\text{loc}}^{1,\infty}$ on every hyperplane $x_n = \text{constant}$. By the theory of Sobolev spaces and Rademacher's theorem (see, for instance, [EG]), g is differentiable in the ordinary sense with respect to the first $n-1$ variables and its weak partials equal its ordinary partials. It follows that g is continuously differentiable with respect to the first $n-1$ variables in the ordinary sense. This completes the proof of both (a) and (b).

(c) We have:

$$\begin{aligned}
u(p(x)) &= \eta(X_{p(x)}) \\
&= \tilde{\eta}(\tilde{X}_x) \\
&= d^{cs}(\log g)(\tilde{X}_x) \\
&= \tilde{X}_x(\log g).
\end{aligned} \tag{2}$$

Note that (2) follows from (b). The second equality of part (c) is an easy consequence of (1).

(d) Let T be a deck transformation of the covering $p : \mathbf{R}^n \rightarrow M$. Since T preserves the foliation $\tilde{W}^{cs} : x_n = \text{constant}$, it follows that its n^{th} component, T_n , depends only on x_n . Note that $T^*\tilde{\omega} = \tilde{\omega}$ and $T^*(dx_n) = T'_n(x_n) dx_n$. Part (d) now easily follows by applying the pullback T^* to the relation $\tilde{\omega} = g dx_n$.

This completes the proof of the theorem. \square

Since $u > 0$, we see that g strictly increases along the orbits of the lifted flow, so g is a global Lyapunov function for \tilde{f}_t . To strengthen this, we will need another result from [S2]. Namely, there we showed that every codimension one Anosov flow can be “synchronized”, i.e. reparametrized by a C^1 function (actually, u) so that for every t , the determinant of the time t map of the synchronized flow f'_t on its strong unstable bundle is identically equal to e^t . Since the center stable bundles of f_t and f'_t are identical, Theorem 1 implies

$$g(\tilde{f}'_t x) \equiv e^t g(x),$$

where \tilde{f}'_t is the lift of the synchronized flow. Now assume $y \in \tilde{W}^{ss}(x)$, where \tilde{W}^{ss} denotes the lift by p of the strong stable foliation of the synchronized flow. Let γ be a path in $\tilde{W}^{ss}(x)$ connecting x to y . Without loss of generality we can assume that x and y are so close that the restriction of p to γ is one-to-one. Then we have:

$$\begin{aligned} |\log g(x) - \log g(y)| &= |\log g(\tilde{f}'_t x) - \log g(\tilde{f}'_t y)| \\ &= \left| \int_{\tilde{f}'_t(\gamma)} \tilde{\eta} \right| \\ &= \left| \int_{f'_t(p(\gamma))} \eta \right| \\ &\leq \|\eta\|_\infty \ell(f'_t(p(\gamma))) \\ &\rightarrow 0, \end{aligned} \tag{3}$$

as $t \rightarrow \infty$. Here ℓ denotes arclength. Note that (3) is just a change of variables formula. Thus $g(x) = g(y)$. Therefore, the foliation by the level sets of g of each leaf of the lifted center stable foliation for f'_t coincides with the lift of the strong stable foliation of f'_t . Thus we have proved the following result:

Theorem 2 *If a codimension one Anosov flow satisfies condition (L), then its lift to the universal covering space (that is \mathbf{R}^n) admits a continuous global Lyapunov function which strictly increases along its orbits and is constant on the leaves of \tilde{W}^{ss} foliation for the synchronized flow.*

Let \mathcal{G} be the partition of \mathbf{R}^n by the level sets of g . (Note that since g is not necessarily C^1 , we cannot claim that the leaves of \mathcal{G} are smooth.) It is natural to ask whether \mathcal{G} is invariant with respect to the deck transformations T . If so, by projecting \mathcal{G} to M , we would obtain a partition $p(\mathcal{G})$ of M which is invariant with respect to the synchronized flow f_t . That would in turn imply that $p(\mathcal{G})$ is a continuous *foliation* tangent to $E^{ss} \oplus E^{uu}$, hence (by [P11]) the flow would be a suspension. Unfortunately, considering Theorem 1 (d), it does not seem easy to prove that \mathcal{G} is projectable to M .

3 Remarks

1) Plante [Pl1] showed that an Anosov flow has a global cross section if its strong stable and strong unstable foliations are jointly integrable. This means that, locally speaking, the projection from a center stable leaf to a center stable leaf along the strong unstable manifolds maps the strong stable leaves to strong stable leaves. In view of that result, it is easy to see that if the function g is constant on the leaves of the \tilde{W}^{uu} foliation for the synchronized flow, then the flow $\{f_t\}$ that we started with admits a global cross section.

2) T. Barbot [Ba] showed that a sufficient condition for a codimension one Anosov flow (in any dimension) to have a cross section is that every leaf of \tilde{W}^{uu} intersects every leaf of \tilde{W}^{cs} exactly once. We state this condition in terms of the function g . As above, let Y be a continuous vector field in E^{uu} such that $\omega(Y) = 1$, let \tilde{Y} be its lift by p , and denote by $\{\tilde{\phi}_t\}$ the lift of the flow of Y (i.e. the flow of \tilde{Y}). Then we have:

Lemma *A codimension one Anosov flow which satisfies condition (L) has a global cross section if for all $x \in \mathbf{R}^n$,*

$$\int_0^{\pm\infty} \frac{1}{g(\tilde{\phi}_s x)} ds = \pm\infty.$$

PROOF Let x_n denote as before the projection to the n^{th} coordinate in \mathbf{R}^n . Then we have:

$$\begin{aligned} \frac{d}{dt} x_n(\tilde{\phi}_t x) &= dx_n(\tilde{Y}_{\tilde{\phi}_t x}) \\ &= \frac{1}{g(\tilde{\phi}_t x)} \tilde{\omega}(\tilde{Y}) \\ &= \frac{1}{g(\tilde{\phi}_t x)}. \end{aligned}$$

Thus

$$x_n(\tilde{\phi}_t x) = \int_0^t \frac{1}{g(\tilde{\phi}_s x)} ds. \quad (4)$$

Since the condition of Barbot is satisfied if the integral on the right hand side of (4) diverges to $+\infty$ as $t \rightarrow +\infty$, and to $-\infty$ as $t \rightarrow -\infty$, the proof is complete. \square

It is not difficult to see that the divergence of this integral is equivalent to the complete integrability of the vector field $Z = g\tilde{Y}$. In other words, that its flow lines through all points of \mathbf{R}^n are defined for all time. One could hope to prove this by finding the “true meaning” of g in terms of objects in the base manifold M .

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