OSELEDETS REGULARITY FUNCTIONS FOR ANOSOV FLOWS

SLOBODAN N. SIMIĆ

Abstract. Oseledets regularity functions quantify the deviation between the growth associated with a dynamical system along its Lyapunov bundles and the corresponding uniform exponential growth. Precise degree of regularity of these functions is unknown. We show that for every invariant Lyapunov bundle of a volume preserving Anosov flow on a closed smooth Riemannian manifold, the corresponding Oseledets regularity functions are in $L^p(m)$, for some $p > 0$, where m is the probability measure defined by the volume form. We prove an analogous result for essentially bounded cocycles over volume preserving Anosov flows.

1. Introduction

This paper is concerned with the so called Oseledets regularity functions, which naturally arise from the Oseledets Multiplicative Ergodic Theorem and the Birkhoff Ergodic Theorem. We restrict our attention to flows; a similar analysis could be done for diffeomorphisms.

For a smooth flow $\Phi = \{f_t\}$ on a closed Riemannian manifold M and a nonzero vector $v \in T_x M$, the Lyapunov exponent of v is defined by

$$\chi(v) = \lim_{|t| \to \infty} \frac{1}{t} \log \|T_x f_t(v)\|,$$

if the limit exists. Vectors v with the same Lyapunov exponent χ (plus the zero vector) form a linear subspace $E^\chi(x)$ of $T_x M$, called the Lyapunov space of χ. By construction, these spaces form an invariant bundle in the sense that $T_x f_t(E^\chi(x)) = E^\chi(f_t x)$, for all $t \in \mathbb{R}$.

The fundamental properties of Lyapunov spaces are described by the following seminal result.

Oseledets Multiplicative Ergodic Theorem ([Ose68, Rue79, BP01]). *Let $\Phi = \{f_t\}$ be a C^1 flow on a closed Riemannian manifold M. There exists a Φ-invariant set $\mathscr{R} \subset M$ of full measure with respect to any Φ-invariant Borel probability measure μ, such that for every $x \in \mathscr{R}$ there exists a splitting (called the Oseledets splitting)*

$$T_x M = \bigoplus_{i=1}^{\ell(x)} E_i(x),$$

and numbers $\chi_1(x) < \cdots < \chi_{\ell(x)}(x)$ with the following properties:

(a) *The bundles E_i are Φ-invariant,*

$$T_x f_t(E_i(x)) = E_i(f_t x),$$

and depend Borel measurably on x.

Date: August 6, 2009.

2000 *Mathematics Subject Classification.* 37D20, 37D25, 37C40.

Key words and phrases. Lyapunov exponent, Oseledets splitting, regularity function.

(b) *For all* $v \in E_i(x) \setminus \{0\}$,
$$\lim_{|t|\to\infty} \frac{1}{t} \log \|T_x f_t(v)\| = \chi_i(x).$$
The convergence is uniform on the unit sphere in $E_i(x)$.

(c) *For for any* $I, J \subset \{1, \ldots, \ell(x)\}$ *with* $I \cap J = \emptyset$, *the angle function is tempered, i.e.,*
$$\lim_{|t|\to\infty} \frac{1}{t} \log \sphericalangle(T_x f_t(E_I(x)), T_x f_t(E_J(x))) = 0,$$
where $E_I = \bigoplus_{i \in I} E_i$.

(d) *For every* $x \in \mathscr{R}$,
$$\lim_{|t|\to\infty} \frac{1}{t} \log \det T_x f_t = \sum_{i=1}^{\ell(x)} \chi_i(x) \dim E_i(x).$$

(e) *If* Φ *is ergodic with respect to* μ, *then the functions* ℓ *and* χ_i *are* μ*-almost everywhere constant.*

Points $x \in \mathscr{R}$ are called regular. Assume Φ is ergodic with respect to some measure μ, fix $i \in \{1, \ldots, \ell\}$, and set $\chi = \chi_i$ and $E = E_i$. Denote the restriction of Tf_t to E by $T^E f_t$. Since
$$\chi = \lim_{t\to\infty} \frac{1}{t} \log \left\| T_x^E f_t \right\|,$$
for each $x \in \mathscr{R}$, it follows that for every $\varepsilon > 0$,
$$\lim_{t\to\infty} \frac{\left\| T_x^E f_t \right\|}{e^{(\chi+\varepsilon)t}} = 0.$$
Therefore, there exists a constant $C > 0$, depending on x and ε such that $\left\| T_x^E f_t \right\| \leq C e^{(\chi+\varepsilon)t}$, for all $t \geq 0$. It is natural to consider the best such C and thus define the following function.

Definition 1.1. *For a fixed Lyapunov bundle* E *of* Φ *and every* $\varepsilon > 0$, *the* (E, ε)*-Oseledets regularity function* $R_\varepsilon : \mathscr{R} \to \mathbb{R}$ *is defined by*
$$R_\varepsilon(x) = \sup_{t \geq 0} \frac{\left\| T_x^E f_t \right\|}{e^{(\chi+\varepsilon)t}}.$$

The family R_ε is the main focus of this paper. It is not hard to see that each R_ε is Borel measurable (see [BP01] for the case of diffeomorphisms) and that $R_\varepsilon \geq 1$. What more can be said about the R_ε? For instance:

Question 1. *Does* R_ε *lie in some* L^p*-space? What is the best value of* p*?*

A related question can be posed for cocycles. Let $u : M \to \mathbb{R}$ be a Borel measurable function and consider a 1-cocycle over a smooth flow $\Phi = \{f_t\}$ on M:
$$\Delta(x, t) = \exp\left\{ \int_0^t u(f_s x)\, ds \right\}. \tag{1.1}$$
When u is essentially bounded with respect to some measure, we will call such a cocycle essentially bounded.

Assume μ is an invariant Borel probability measure, $u \in L^\infty(\mu)$, and set $\chi = \int_M u\, d\mu$. If μ is ergodic, then by the Birkhoff Ergodic Theorem,
$$\lim_{t\to\infty} \frac{1}{t} \log \Delta(x, t) = \chi,$$

for μ-a.e. x. Denote the set of these Birkhoff regular points by $\mathscr{R}$ as well. Then for every $x \in \mathscr{R}$ and $\varepsilon > 0$, $\Delta(x,t)/\exp\{(\chi+\varepsilon)t\} \to 0$, as $t \to \infty$, so there exists a constant $B > 0$ (depending on x and ε) such that $\Delta(x,t) \leq B\exp\{(\chi+\varepsilon)t\}$, for all $t \geq 0$. It makes sense to consider the best such B and define the following function.

Definition 1.2. *For each $\varepsilon > 0$, the (u,ε)-regularity function $D^u_\varepsilon : M \to \mathbb{R}$ is defined by*

$$D^u_\varepsilon(x) = \sup_{t\geq 0} \frac{\Delta(x,t)}{e^{(\chi+\varepsilon)t}}. \tag{1.2}$$

When u is clear from the context, we will write D_ε. It is clear that each D_ε is Borel measurable and $D_\varepsilon \geq 1$. It is also easy to see that if $\varepsilon \geq \|u\|_\infty - \chi$, then $D_\varepsilon = 1$, μ-a.e. We are therefore interested only in the values of ε less than $\|u\|_\infty - \chi$.

What more can be said about the D_ε? It is natural to ask:

Question 2. *Does D_ε belong to some L^p-space? What is the best value of p?*

A word of caution is in place here. Even for "best" (interesting) dynamical systems, namely globally uniformly hyperbolic ones, neither the Oseledets nor the Birkhoff theorem guarantee any particularly good properties of the set $\mathscr{R}$ of regular points and as a consequence of the regularity functions. Although $\mathscr{R}$ has full measure with respect to any invariant probability measure, its complement is not only non-empty, but can be topologically very large. This follows from the work of Barriera and Schmeling [BS00b] who showed that for an Anosov diffeomorphism of the 2-torus, the complement of the set of regular points can have the full Hausdorff dimension (i.e., two). In the continuous time case, using multifractal analysis, Barreira and Saussol [BS00a] showed that for hyperbolic flows the set of non-regular points is similarly topologically large, namely, dense and of full Hausdorff dimension, for a generic function u. Similar results were obtained by Pesin and Sadovskaya [PS01].

We will soon see that Questions 1 and 2 are closely related at least in the case of Anosov flows, to which we now restrict ourselves. Namely, given a volume preserving Anosov flow Φ and a Lyapunov bundle E of Φ, it turns out that each regularity function R_ε of E can be related to a regularity function D^u_η, for some $\eta > 0$ and some essentially bounded function u dependent only on E. See Theorem B.

We now state our main results. Throughout, m will denote the Borel probability measure defined by the Riemannian metric (and volume form) on M.

Theorem A. *Let $\Phi = \{f_t\}$ be a C^2 volume preserving Anosov flow on a closed Riemannian manifold M and let $\Delta : M \times \mathbb{R} \to \mathbb{R}$ be a cocycle over Φ, defined as in (1.1). If $u \in L^\infty(m)$, then for every $\varepsilon > 0$, the corresponding (u,ε)-regularity function D_ε belongs to $L^p(m)$, for some $p > 0$.*

If u is Hölder continuous, let H be the entropy function of u (as defined in § 2.1). Then $D_\varepsilon \in L^p(m)$, provided that

$$p \leq \left\{ \int_\varepsilon^{\|u\|_\infty - \chi} \frac{ds}{H(\chi+s)} \right\}^{-1}. \tag{1.3}$$

Here is a sketch of the proof of Theorem A. If u is Hölder, then for all $x \in \mathscr{R}$, $t \mapsto \Delta(x,t)$ is continuous, so we define $T_\varepsilon : \mathscr{R} \to \mathbb{R}$ $(0 < \varepsilon < \|u\|_\infty - \chi)$ to be the smallest $t \geq 0$ at which the supremum in (1.2) is attained. Then T_ε is Borel measurable and $D_\varepsilon \leq \exp\{(\|u\|_\infty - \chi - \varepsilon)T_\varepsilon\}$, so we study the question of integrability of $\exp T_\varepsilon$. Using a large deviations result of Waddington [Wad96] (see § 2.1 for details), we show that if $p < H(\chi + \varepsilon)$, then $\exp T_\varepsilon \in L^p(m)$, where H is the entropy function of u. Next, we show that if $\eta < \varepsilon$, then $D_\eta \leq D_\varepsilon \exp\{(\varepsilon - \eta)T_\eta\}$ m-a.e., which for any natural number N by induction extends to

$$D_\varepsilon \leq \prod_{i=0}^{N-1} \exp(\delta T_{\varepsilon_i}),$$

where $\varepsilon = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_N = \|u\|_\infty - \chi$ is a partition of $[\varepsilon, \|u\|_\infty - \chi]$ with $\delta = \varepsilon_{i+1} - \varepsilon_i = (\|u\|_\infty - \chi - \varepsilon)/N$, for all i. Using the generalized Hölder inequality and the fact that $\exp(\delta T_{\varepsilon_i}) \in L^{p_i}(m)$, where $p_i < H(\chi + \varepsilon_i)/\delta$, we obtain $D_\varepsilon \in L^p(m)$, where $p^{-1} = \sum_i p_i^{-1} > \sum_i \delta / H(\chi + \varepsilon_i)$. Passing to the limit as $N \to \infty$ in the last sum, we obtain (1.3).

If u is only essentially bounded, we show that it is possible to suitably approximate u by a larger smooth function (Lemma 3.3). Namely, for every $\delta > 0$ there exists a C^∞ function $\tilde{u} : M \to \mathbb{R}$ such that $u \leq \tilde{u}$ and $\int_M (\tilde{u} - u)\, dm < \delta$. It then easily follows that for any $0 < \delta < \varepsilon < \|u\|_\infty - \chi$, $D_\varepsilon^u \leq D_{\varepsilon-\delta}^{\tilde{u}}$, m-a.e., which implies that D_ε^u lies in some L^p space.

A bridge between the two different types of regularity functions is given by the following result.

Theorem B. *Let $\Phi = \{f_t\}$ be a C^2 volume preserving Anosov flow on a closed C^∞ Riemannian manifold M and let E be a Lyapunov bundle for Φ associated with a Lyapunov exponent χ. For every $\delta > 0$ there exists a constant $C_\delta > 0$ such that*

$$\left\| T_x^E f_t \right\| \leq C_\delta e^{\delta t} \exp\left\{ \int_0^t u(f_s x)\, ds \right\},$$

for every $x \in \mathscr{R}$ and $t \geq 0$, where $u \in L^\infty(m)$ is independent of δ and

$$\int_M u\, dm = \chi.$$

The proof of Theorem B goes as follows. First, we trivialize E by using a measurable orthonormal frame. This transforms the second variational equation for the restriction of the flow to E into a family of non-autonomous differential equations $\dot{X} = A_x(t)X$ on $\mathbb{R}^k$ ($k = \dim E$), parametrized by $x \in \mathscr{R}$. Following [BP01], we use a lemma of Perron to construct a family $U_x(t)$ of orthogonal matrices such that if $v(t)$ is a solution to $\dot{v} = A_x(t)v$, then $z(t) = U_x(t)v(t)$ is a solution to $\dot{z} = B_x(t)z$, where $B_x(t)$ are upper triangular matrices, whose non-diagonal entries are bounded in x and t. We show that for every $\delta > 0$ there exists a norm $\|\cdot\|_\delta$ on $\mathbb{R}^k$ such that $\|B_x(t)\|_\delta < r(B_x(t)) + \delta$, where r denotes the spectral radius. Furthermore, $r(B_x(s+t)) = r(B_{f_s x}(t))$, for all t and a.e. x. We then show that if $X(t)$ is the unique solution to the matrix differential equation $\dot{X} = A_x(t)X$ satisfying $X(0) = I$, then

$$\|X(t)\|_\delta \leq K_\delta e^{\delta t} \exp\left\{ \int_0^t r(B_x(s))\, ds \right\},$$

for some constant $K_\delta > 0$. We therefore define $u : M \to \mathbb{R}$ by $u(x) = r(B_x(0))$. It is not hard to prove that u is essentially bounded. The desired inequality for $T_x^E f_t$ is now obtained by pulling the norms $\|\cdot\|_\delta$ back to E and observing that the each new Finsler structure is globally uniformly equivalent to the original one.

The last result is a straightforward corollary of Theorem B.

Theorem C. *Let $\Phi = \{f_t\}$ be a C^2 volume preserving Anosov flow on a closed C^∞ Riemannian manifold M. Let E be a Lyapunov bundle in the Oseledets splitting for Φ. Then for* *every $\varepsilon > 0$, the corresponding (E, ε)-regularity function R_ε belongs to the space $L^p(m)$, for some $p > 0$.*

To prove Theorem C, denote the Lyapunov exponent corresponding to E by χ and let $\varepsilon > 0$ and $0 < \delta < \varepsilon$ be arbitrary. Then by Theorem B,

$$\begin{aligned}\frac{\|T_x^E f_t\|}{e^{(\chi+\varepsilon)t}} &\le C_\delta \frac{\exp\left\{\int_0^t u(f_s x)\, ds\right\}}{e^{(\chi+\varepsilon-\delta)t}} \\ &\le C_\delta D_{\varepsilon-\delta}(x),\end{aligned}$$

for m-a.e. $x \in \mathscr{R}$ and $t \ge 0$. This implies that $R_\varepsilon \le C_\delta D_{\varepsilon-\delta}$, which yields Theorem C.

Remark. The question of the best $p = p(\varepsilon)$ such that $D_\varepsilon \in L^p(m)$ (and the analogous question for R_ε) remains open. It is likely that the answer can be found by a more careful analysis of the set $\mathscr{L} = \{(\varepsilon, p) : D_\varepsilon \in L^p(m)\}$. This set possesses a number of interesting properties such as: (a) $\{(\varepsilon, p^{-1}) : (\varepsilon, p) \in \mathscr{L}\}$ is convex; (b) if ν is a Borel probability measure on an interval $I \subset (0, \|u\|_\infty - \chi)$ and $\phi : I \to \mathbb{R}$ a positive Borel function whose graph is contained in $\mathscr{L}$, then $(\varepsilon, p) \in \mathscr{L}$, where $\varepsilon = \int_I t\, d\nu(t)$ and $p = \left(\int_I d\nu/\phi\right)^{-1}$.

The paper is organized as follows. In Section 2 we recall some basics facts about Anosov flows, present the large deviation result of Waddington [Wad96], and review some Pesin-Lyapunov theory used later in the paper. Theorem A is proved in Section 3 and Theorem B in Section 4.

2. Preliminaries

2.1. **Large deviations for Anosov flows.** A non-singular C^1 flow $\Phi = \{f_t\}$ on a closed Riemannian manifold M is called an Anosov flow if there exists a Tf_t-invariant continuous splitting of the tangent bundle,

$$TM = E^{uu} \oplus E^c \oplus E^{ss},$$

and constants $C, \lambda > 0$ such that for all $t \ge 0$,

$$\|Tf_t \restriction_{E^{ss}}\| \le Ce^{-\lambda t} \qquad \text{and} \qquad \|Tf_t \restriction_{E^{uu}}\| \ge Ce^{\lambda t},$$

where the center bundle E^c is one dimensional and generated by the infinitesimal generator of the flow. The bundles E^{uu}, E^{ss} are called the strong unstable and strong stable bundle of the flow. If the flow is of class C^2, E^{ss}, E^{uu} are known to be Hölder continuous (cf., [Has94, Has97, HPS77]). If an Anosov flow preserves a volume form, it is automatically ergodic with respect to the Lebesgue measure defined by the volume (see [Ano67]). Recall that a flow is called (topologically) transitive if it has a dense orbit.

An equilibrium state of a function $\varphi : M \to \mathbb{R}$ is an invariant Borel probability measure μ at which the quantity

$$h(\mu) + \int_M \varphi\, d\mu$$

attains its supremum, where $h(\mu)$ denotes the measure theoretic entropy of φ with respect to μ. This supremum $P(\varphi)$ is called the pressure of φ. If φ is Hölder continuous, there exists a *unique* equilibrium state of φ, denoted by μ_φ.

Given a transitive Anosov flow $\Phi = \{f_t\}$, one defines a function $\varphi^u : M \to \mathbb{R}$ by

$$\varphi^u(x) = \left.\frac{d}{dt}\right|_0 \log\det T_x f_t\restriction_{E^{uu}} .$$

If Φ is C^2, this function is known to be Hölder continuous. The unique equilibrium state of $-\varphi^u$ is called the Sinai-Ruelle-Bowen (SRB) measure μ_{SRB} of φ. By the Bowen-Ruelle theorem [BR75], for every continuous $\varphi : M \to \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(f_t x)\, dt = \int_M \varphi\, d\mu_{\mathrm{SRB}},$$

for m-a.e. $x \in M$, where m is the Lebesgue measure defined by the volume form and Φ is C^2. Thus μ_{SRB} is an ergodic measure for Φ. If the flow admits a smooth invariant Borel probability measure μ (i.e., a measure that is absolutely continuous with respect to the volume measure m), then by the Birkhoff ergodic theorem, $\mu = \mu_{\mathrm{SRB}}$. In particular, if Φ is volume preserving, then $\mu_{\mathrm{SRB}} = m$.

For an arbitrary flow $f_t : M \to M$ and function $\psi : M \to \mathbb{R}$, we can define a skew product flow

$$S_t^\psi : S^1 \times M \to S^1 \times M$$

by

$$S_t^h(\exp(2\pi i\theta), x) = (\exp(2\pi i(\theta + \psi^t(x)), f_t(x)),$$

where

$$\psi^t(x) = \int_0^t \psi(f_s x)\, ds.$$

Definition 2.1 ([Wad96]). *A Hölder continuous function $\varphi : M \to \mathbb{R}$ and a flow $\Phi = \{f_t\}$ on M are called flow independent if they satisfy the following property: for every two numbers $a, b \in \mathbb{R}$, if the skew product flow $S_t^{a+b\varphi}$ is* not *topologically transitive*[1]*, then $a = 0 = b$.*

Large deviation asymptotics for transitive Anosov flows were established by Waddington in [Wad96]. In particular:

Theorem 2.2 (Corollary 2, [Wad96]). *Let $\Phi = \{f_t\}$ be a transitive C^2 Anosov flow on M and let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function such that φ and Φ are flow independent. There exist analytic functions $\beta, \gamma, \rho : \mathbb{R} \to \mathbb{R}$, such that if $\rho(a) > 0$, then*

$$\mu_{\mathrm{SRB}}\left\{x : \int_0^T \varphi(f_t x)\, dt \geq Ta\right\} \sim \frac{C(a)}{\rho(a)} \frac{1}{2\pi\beta''(\rho(a))} \frac{e^{\gamma(a)T}}{\sqrt{T}},$$

as $T \to \infty$, where $C(a)$ is a constant depending on a.

Here, $a(t) \sim b(t)$ as $t \to \infty$, means $a(t)/b(t) \to 1$. The function $\beta : \mathbb{R} \to \mathbb{R}$ is defined by $\beta(t) = P(\psi + t\varphi) - P(\psi)$, for a Hölder continuous ψ. For our purposes, we will take $\psi = 0$. Note that if $\psi = 0$, then $\mu_\psi = \mu_0$ is the measure of maximal entropy for Φ, and if the flow is volume preserving, then $\mu_0 = m$.

Some properties of β (see [Wad96] for details), with $\psi = 0$, are:

$$\beta'(t) = \int_M \varphi\, d\mu_{t\varphi} \quad \text{and} \quad \beta''(t) = \sigma^2_{\mu_{t\varphi}}(\varphi), \tag{2.1}$$

[1] Waddington uses the term topologically ergodic, which has the same meaning, see [Pet89], Proposition 2.4.

where for a measure μ with $\int_M \varphi\, d\mu = \chi$, the variance of φ is defined by

$$\sigma_\mu^2(\varphi) = \lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \varphi\circ f_t\, dt - \chi T\right)^2.$$

Furthermore, $\sigma_\mu^2(\varphi) = 0$ if and only if φ is cohomologous to a constant. If φ is not cohomologous to a constant, the map $t \mapsto \beta'(t)$ is strictly increasing. Denote its range by Γ_φ; it follows from (2.1) that $\Gamma_\varphi \subset (\min\varphi, \max\varphi)$. On Γ_φ, set $\rho = (\beta')^{-1}$. Then $\rho : \Gamma_\varphi \to \mathbb{R}$ is strictly increasing, surjective, and real analytic, with $\rho(\chi) = 0$. Finally, $\gamma : \Gamma_\varphi \to \mathbb{R}$ is defined as minus one times the Legendre transform of β, i.e.,

$$\gamma(s) = -\sup_{t\in\mathbb{R}}\{st - \beta(t)\}.$$

It can be shown that γ is a strictly concave, non-positive function with a unique maximum at $\chi = \int_M \varphi\, dm$ (where we still take $\psi = 0$). Furthermore, $\gamma''(s) = -1/\beta''(\rho(s))$, so in particular, $\gamma''(\chi) = -1/\sigma_m^2(\varphi)$ (cf., [Wad96]).

In the large deviations literature the function $H = -\gamma$ is called the entropy function of φ. It is easily seen that H has the following properties (see [Wad96]): it is strictly convex on Γ_φ,

$$H(\chi) = H'(\chi) = 0, \quad H''(\chi) = \frac{1}{\sigma_m^2(\varphi)}, \quad \text{and} \quad H(a) = \infty \text{ for } a \notin \Gamma_\varphi,$$

where $\chi = \int_M \varphi\, dm$.

The following lemma will be needed later in the paper.

Lemma 2.3. *Let Φ be a volume preserving Anosov flow and $\varphi : M \to \mathbb{R}$ a Hölder continuous function. If φ and Φ are not flow independent, then φ is cohomologous to a constant.*

Proof. Suppose φ and Φ are not flow independent. Then there exist numbers a, b, not both zero, such that the skew product $S_t^{a+b\varphi}$ is not topologically transitive. The skew product is clearly not ergodic with respect to the measure $m_1 \times m$, where m_1 is the Haar-Lebesgue measure on S^1. Since the volume measure is an equilibrium state of Φ, Proposition 4.2 in [Wal99] implies the existence of a nonzero integer ℓ and a Hölder function $w : M \to \mathbb{R}$ such that

$$\ell \int_0^t (a + b\varphi)(f_s x) = w(f_t x) - w(x),$$

for all $x \in M$. If $b = 0$, then $a \neq 0$ and $w(f_t x) - w(x) = a\ell t$ everywhere, which is impossible. Therefore, $b \neq 0$. Differentiating the above identity, we obtain

$$\varphi + \frac{a}{b} = \frac{1}{\ell} Xw,$$

which means that φ is cohomologous to $-a/b$. □

2.2. **Pesin-Lyapunov theory.** In this section we follow Barreira-Pesin [BP01] and briefly review some elements of Pesin-Lyapunov theory for linear differential equations

$$\dot v = A(t)v, \tag{2.2}$$

where $A(t)$ is a $k \times k$ bounded matrix function, i.e.,

$$\sup_{t\in\mathbb{R}} \|A(t)\| < \infty.$$

We concentrate on *real* matrices $A(t)$ ([BP01] deals with complex matrices). The Lyapunov exponent of $v \in \mathbb{R}^k$ is the number

$$\chi(v) = \limsup_{t\to\infty} \frac{1}{t} \log \|v(t)\|,$$

where $v(t)$ is the unique solution to (2.2) satisfying the initial condition $v(0) = v$. The function $\chi : \mathbb{R}^k \to \mathbb{R} \cup \{-\infty\}$ attains only finitely many values $\chi_1 < \ldots < \chi_\ell$, where $\ell \le k$. For each $1 \le i \le \ell$, define

$$V_i = \{v \in \mathbb{R}^k : \chi(v) \le \chi_i\}.$$

This defines a linear filtration of $\mathbb{R}^k$:

$$\{\mathbf{0}\} = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_\ell = \mathbb{R}^k.$$

An ordered basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$ is called normal with respect to the filtration $\mathscr{V} = \{V_i\}$ if for every $1 \le i \le \ell$, the vectors $v_1, \ldots, v_{k_i}$ form a basis for V_i, where $k_i = \dim V_i$. In particular, if χ is a constant function, every basis of $\mathbb{R}^k$ is normal.

Given a basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$ and $1 \le m \le k$, denote by $\Gamma_m^{\mathbf{v}}(t)$ the volume of the parallelepiped defined by $v_1(t), \ldots, v_m(t)$, where $v_i(t)$ is the unique solution to (2.2) satisfying $v_i(0) = v_i$. Recall that the Lyapunov exponent χ is regular (together with the Lyapunov exponent $\tilde{\chi}$ associated with the dual equation $\dot{w} = -A(t)^* w$) if and only if (see [BP01], Theorem 1.3.1) for any normal ordered basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$ and $1 \le m \le k$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \Gamma_m^{\mathbf{v}}(t) = \sum_{i=1}^{m} \chi(v_i).$$

In particular, if χ is constant, then for any basis $\mathbf{v} = (v_1, \ldots, v_k)$ and $1 \le m \le k$,

$$\lim_{t\to\infty} \frac{1}{t} \log \Gamma_m^{\mathbf{v}}(t) = m\chi. \tag{2.3}$$

Let us recall how one converts (as in [BP01]) by a linear change of coordinates the equation (2.2) into $\dot{z} = B(t)z$, where the matrix $B(t)$ is *upper triangular*. We seek a differentiable family of orthogonal matrices $U(t)$. Set $z(t) = U(t)^{-1}v(t)$, where $v(t)$ is a solution to (2.2); then

$$\dot{v}(t) = \dot{U}(t)z(t) + U(t)\dot{z}(t) = A(t)U(t)z(t),$$

which implies $\dot{z}(t) = B(t)z(t)$, where

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}\dot{U}(t). \tag{2.4}$$

The following lemma of Perron guarantees the existence of $U(t)$ so that $B(t)$ is upper triangular, for all t.

Lemma 2.4 (Lemma 1.3.3, [BP01]). *There exists a differentiable matrix function $t \mapsto U(t)$ such that for each $t \ge 0$:*

(a) *$U(t)$ is unitary.*
(b) *$B(t) = [b_{ij}(t)]$ is upper triangular.*
(c) *For all $1 \le i < j \le k$,*

$$\sup_{t\ge 0} |b_{ij}(t)| \le 2 \sup_{t\ge 0} \|A(t)\| < \infty.$$

(d) *For any basis* $\mathbf{v} = (v_1, \ldots, v_k)$ *of* $\mathbb{R}^k$ *and all* $1 \le i \le k$*,*

$$b_{ii}(t) = \frac{d}{dt} \log \frac{\Gamma_i^{\mathbf{v}}(t)}{\Gamma_{i-1}^{\mathbf{v}}(t)}.$$

Here is how families $U(t)$ and $B(t)$ are constructed. Denote by

$$\mathscr{G} : Gl(k, \mathbb{R}) \to O(k)$$

the Gram-Schmidt orthogonalization operator that sends a basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$ to an orthonormal basis $\mathbf{u} = (u_1, \ldots, u_k)$. We can think of $\mathbf{v}$ and $\mathbf{u}$ as matrices with columns $v_1, \ldots, v_k$ and $u_1, \ldots, u_k$, respectively. Then $\mathbf{v} \in Gl(k, \mathbb{R})$ and $\mathbf{u} \in O(k)$. Observe that

$$\mathscr{G} = \mathscr{N} \circ \mathscr{L},$$

where $\mathscr{L}[v_1, \ldots, v_k] = [w_1, \ldots, w_k]$ is a linear operator defined by

$$w_{i+1} = v_{i+1} - \mathrm{proj}_{W_i} v_{i+1}, \qquad W_i = \mathrm{span}\{w_1, \ldots, w_i\},$$

and $\mathscr{N}[w_1, \ldots, w_k] = [u_1, \ldots, u_k]$ is the normalization operator

$$u_i = \frac{w_i}{\|w_i\|}.$$

Since $\mathscr{L}$ is linear, differentiating $\mathscr{G}$ at v yields

$$T_{\mathbf{v}}\mathscr{G} = T_{\mathscr{L}\mathbf{v}}\mathscr{N} \circ \mathscr{L}.$$

In the proof of Perron's Lemma 1.3.1 in [BP01], for an arbitrary but fixed basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$, one defines $U(t)$ by

$$U(t) = \mathscr{G}[v_1(t), \ldots, v_k(t)],$$

where $v_i(t)$ is the unique solution to the equation $\dot{v} = A(t)v$ satisfying the initial condition $v(0) = v_i$.

The family $B(t)$ is defined as in (2.4). Thus both $t \mapsto U(t)$ and $t \mapsto B(t)$ depend on the choice of a basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathbb{R}^k$. When it is important to emphasize this, we will write $U^{\mathbf{v}}(t)$ and $B^{\mathbf{v}}(t)$.

It is clear that the eigenvalues of $B(t)$ are its diagonal entries $b_{ii}(t)$. Denote the spectral radius of a matrix M by $r(M)$.

Corollary 2.5. *If* χ *is constant, then*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t r(B(s))\, ds = \chi.$$

Proof. Follows directly from (2.3) and part (d) of Lemma 2.4. □

Lemma 2.6. *The spectral radius* $r(B(t))$ *of the matrix* $B(t) = B^{\mathbf{v}}(t)$ *is independent of the choice of the basis* $\mathbf{v} = (v_1, \ldots, v_k)$ *of* $\mathbb{R}^k$*.*

Proof. Let $\mathbf{v} = (v_1, \ldots, v_k)$ and $\mathbf{w} = (w_1, \ldots, w_k)$ be two bases of $\mathbb{R}^k$ and let $B^{\mathbf{v}}(t)$ and $B^{\mathbf{w}}(t)$ be the corresponding matrices constructed as above. Denote the solutions to (2.2) with initial values $(v_1, \ldots, v_k)$, $(w_1, \ldots, w_k)$ by $(v_1(t), \ldots, v_k(t))$ and $(w_1(t), \ldots, w_k(t))$, respectively. Both k-tuples are bases of $\mathbb{R}^k$. Therefore, there exists a family of invertible matrices $P(t)$ such that $P(t)v_i(t) = w_i(t)$, for all $1 \le i \le k$. It follows that

$$\Gamma_i^{\mathbf{w}}(t) = \det P(t)\, \Gamma_i^{\mathbf{v}}(t),$$

for all $1 \leq i \leq k$, and thus

$$\frac{\Gamma_i^{\mathbf{w}}(t)}{\Gamma_{i-1}^{\mathbf{w}}(t)} = \frac{\Gamma_i^{\mathbf{v}}(t)}{\Gamma_{i-1}^{\mathbf{v}}(t)},$$

for all $t \geq 0$. Lemma 2.4(d) implies that the corresponding diagonal entries of $B^{\mathbf{v}}(t)$ and $B^{\mathbf{w}}(t)$ are the same, which yields the conclusion of the lemma. □

We define a function $\rho_B : \mathbb{R} \to \mathbb{R}$ by

$$\rho_B(t) = r(B(t)).$$

Lemma 2.7. *There exists a universal constant $K > 0$, depending only on k, such that*

$$|\rho_B(0)| \leq K \|A(0)\|.$$

Proof. Let $\mathbf{v} = \mathbf{e}$ be the standard basis $(e_1, \ldots, e_k)$ of $\mathbb{R}^k$ and let $U(t) = U^{\mathbf{e}}(t)$ be the corresponding orthogonal matrix function defined as above. Then:

$$\begin{aligned}
|\rho_B(0)| &= |r(B(0))| \\
&= \left| r(A(0) - U(0)^{-1}\dot{U}(0)) \right| \\
&\leq \left\| A(0) - U(0)^{-1}\dot{U}(0) \right\| \\
&\leq \|A(0)\| + \left\| U(0)^{-1}\dot{U}(0) \right\| \\
&= \|A(0)\| + \left\| \dot{U}(0) \right\|.
\end{aligned}$$

Denote the solution to (2.2) with initial value e_i by $e_i(t)$. Then:

$$\begin{aligned}
\dot{U}(0) &= \left.\frac{d}{dt}\right|_0 \mathscr{G}[e_1(t), \ldots, e_k(t)] \\
&= T_I\mathscr{G}[\dot{e}_1(0), \ldots, \dot{e}_k(0)] \\
&= T_I\mathscr{G}[A(0)e_1, \ldots, A(0)e_k] \\
&= T_I\mathscr{G}(A(0)),
\end{aligned}$$

where I is the $k \times k$ identity matrix. Let $K = 1 + \|T_I\mathscr{G}\|$, where $T_I\mathscr{G}$ is regarded as a map between Lie algebras $\mathfrak{gl}_k$ and $\mathfrak{o}_k$. It follows that

$$\rho_B(0) \leq K \|A(0)\|,$$

completing the proof of the lemma. □

3. Proof of Theorem A

We split the proof of Theorem A into two cases: in the first case, we deal with Hölder continuous functions u. The general case of essentially bounded u is reduced to the first case in a suitable way.

Case 1: *u is Hölder continuous.*

First of all, we may assume that u and Φ are flow independent. Otherwise, by Lemma 2.3, u is cohomologous to a constant, which is necessarily equal to $\chi = \int_M u\,dm$, that is, $u = Xw+\chi$, for some Hölder function w. This implies that

$$\exp\left\{\int_0^t u(f_s x)\,ds\right\} = e^{\chi t} e^{w(f_t x)-w(x)} \leq e^{2\|w\|_\infty} e^{\chi t},$$

so the corresponding regularity functions D_ε are all constant (in fact, $D_\varepsilon = 1$ m-a.e., for all $\varepsilon > 0$).

Recall that we are only interested in the values $0 < \varepsilon < \|u\|_\infty - \chi$, since $D_\varepsilon = 1$ for $\varepsilon \geq \|u\|_\infty - \chi$.

Denote the set of Birkhoff regular points by $\mathscr{R}$. For each $x \in \mathscr{R}$ and $0 < \varepsilon < \|u\|_\infty - \chi$, define $T_\varepsilon(x)$ to be the smallest $T \geq 0$ at which the supremum defining $D_\varepsilon = D_\varepsilon^u$ in (1.2) is attained. That is, let

$$T_\varepsilon(x) = \min\left\{T \geq 0 : \int_0^T u(f_s x)\,ds - (\chi+\varepsilon)T = \log D_\varepsilon(x)\right\}.$$

By the Birkhoff Ergodic Theorem, $T_\varepsilon : \mathscr{R} \to [0,\infty)$ is well-defined. It is clear that T_ε is Borel measurable.

As in § 2.1, let $H = -\gamma$ be the entropy function of u.

Lemma 3.1. *If $p < H(\chi+\varepsilon)$, then $e^{T_\varepsilon} \in L^p(m)$.*

Proof. Fix an $\varepsilon \in (0, \|u\|_\infty - \chi)$ and let $\zeta > 1$ be arbitrary. Define

$$B_n = \{x : \zeta^n < T_\varepsilon(x) \leq \zeta^{n+1}\}.$$

Suppose $x \in B_n$. Then

$$\begin{aligned}
\int_0^{\zeta^{n+1}} u(f_s x)\,ds &\geq \int_0^{T\varepsilon(x)} u(f_s x)\,ds \\
&= (\chi+\varepsilon)T_\varepsilon(x) + \log D_\varepsilon(x) \\
&\geq (\chi+\varepsilon)T_\varepsilon(x) \\
&\geq (\chi+\varepsilon)\zeta^n \\
&= \frac{\chi+\varepsilon}{\zeta}\zeta^{n+1}.
\end{aligned}$$

By Theorem 2.2 there exists a constant L depending on ε and ζ such that

$$m(B_n) \leq L\exp\left\{-H\left(\frac{\chi+\varepsilon}{\zeta}\right)\zeta^{n+1}\right\}.$$

It follows that

$$\begin{aligned}
\int_M \exp(pT_\varepsilon)\,dm &= \sum_{n=0}^\infty \int_{B_n} \exp(pT_\varepsilon)\,dm \\
&\leq \sum_n L\exp\left\{p\zeta^{n+1} - H\left(\frac{\chi+\varepsilon}{\zeta}\right)\zeta^{n+1}\right\},
\end{aligned}$$

which is finite for

$$p < H\left(\frac{\chi+\varepsilon}{\zeta}\right).$$

Since $\zeta > 1$ was arbitrary, letting $\zeta \to 1+$ yields the claim. $\square$

Lemma 3.2. *If $0 < \eta < \varepsilon$ and $x \in \mathscr{R}$, then*

$$D_\eta(x) \le D_\varepsilon(x) e^{(\varepsilon-\eta)T_\eta(x)}.$$

Proof. Set $u_\varepsilon = u - \chi - \varepsilon$. Then for each $x \in \mathscr{R}$:

$$\begin{aligned}
\log D_\varepsilon(x) &= \max_{t\ge 0} \int_0^t u_\varepsilon(f_s x)\, ds \\
&\ge \int_0^{T_\eta(x)} u_\varepsilon(f_s x)\, ds \\
&= \int_0^{T_\eta(x)} u_\eta(f_s x)\, ds + (\eta - \varepsilon) T_\eta(x) \\
&= \log D_\eta(x) - (\varepsilon - \eta) T_\eta(x),
\end{aligned}$$

which proves the claim. □

Now let $0 < \varepsilon < \|u\|_\infty - \chi$ be arbitrary and fix a natural number $N \ge 1$. Let

$$\varepsilon = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_N = \|u\|_\infty - \chi$$

be a partition of the interval $[\varepsilon, \|u\|_\infty - \chi]$ with $\delta = \varepsilon_{i+1} - \varepsilon_i = (\|u\|_\infty - \chi - \varepsilon)/N$, for all $0 \le i \le N-1$. Applying Lemma 3.2 repeatedly and using $D_{\|u\|_\infty - \chi} = 1$ a.e., we obtain

$$D_\varepsilon \le \prod_{i=0}^{N-1} \exp(\delta T_{\varepsilon_i}).$$

Since $e^{\delta T_{\varepsilon_i}} \in L^{p_i}(m)$, for all $p_i < H(\chi + \varepsilon_i)/\delta$ (Lemma 3.1), the generalized Hölder inequality yields $D_\varepsilon \in L^p(m)$, where

$$\begin{aligned}
\frac{1}{p} &= \sum_{i=0}^{N-1} \frac{1}{p_i} \\
&> \sum_{i=0}^{N-1} \frac{\delta}{H(\chi + \varepsilon_i)} \\
&\to \int_\varepsilon^{\|u\|_\infty - \chi} \frac{ds}{H(\chi + s)},
\end{aligned}$$

as $N \to \infty$. This proves the second conclusion of Theorem A.

Remark. It is possible to show that, in fact,

$$D_\varepsilon(x) = \exp\left\{ \int_\varepsilon^{\|u\|_\infty - \chi} T_\eta(x)\, d\eta \right\},$$

for m-almost every $x \in M$.

Case 2: $u \in L^\infty(m)$.

We need to show that for every $\varepsilon > 0$ there exists $p > 0$ such that $D_\varepsilon \in L^p(m)$. The following lemma says that we may as well work with a smooth u.

Lemma 3.3. *For every $\delta > 0$ there exists a C^∞ function $\tilde{u} : M \to \mathbb{R}$ such that $u \leq \tilde{u}$ and*

$$\int_M (\tilde{u} - u)\, dm < \delta.$$

Proof. We will first find a continuous function $w : M \to \mathbb{R}$ such that $u \leq w$ and $\int (w - u) < \delta$ and then regularize w.

Let $\eta > 0$ be arbitrary. By Luzin's theorem, there exists a continuous function $g : M \to [0, \infty)$ such that $\|g\|_\infty \leq \|u\|_\infty$ and $m(A) < \eta$, where $A = \{x \in M : u(x) \neq g(x)\}$. The set A is Borel measurable, so there exists an open set U such that $A \subset U$ and $m(U \setminus A) < \eta$. Let V be an open set such that

$$A \subset V \subset \overline{V} \subset U.$$

By Urysohn's lemma, there exists a continuous function $h : M \to [0, 1]$ such that $h = 0$ on the complement of U and $h = 1$ on $\overline{V}$. Let $k \in (\|u\|_\infty, 2\|u\|_\infty)$ and define

$$w = g + kh.$$

On A, $w \geq kh = k > u$. On the complement of A, $g = u$, so $w = u + kh \geq u$. Observe that $w = u$ on the complement of U and that $g + kh \leq 3\|u\|_\infty$ on U. Therefore,

$$\begin{aligned} \int_M (w - u)\, dm &= \int_U (g + kh - u)\, dm \\ &\leq \int_U (g + kh)\, dm \\ &\leq m(U) \cdot 3\|u\|_\infty \\ &\leq 6\eta \|u\|_\infty. \end{aligned}$$

Let $w_a = w + a$, where $a > 0$ is a small constant, so that $w_a - u \geq a$. Finally, let $\tilde{u}$ be a C^∞ regularization of w_a with $\|\tilde{u} - w_a\|_\infty$ sufficiently small so that $\tilde{u} \geq u$. It is easy to see that the integrals of $\tilde{u}$ and w_a are the same. Since

$$\int_M (\tilde{u} - u)\, dm = \int_M (w_a - u)\, dm \leq 6\eta \|u\|_\infty + a,$$

by choosing η and a sufficiently small, we obtain a desired function $\tilde{u}$. □

Fix an $\varepsilon \in (0, \|u\|_\infty - \chi)$ and $0 < \delta < \varepsilon$. Let $\tilde{u}$ be a C^∞ function on M supplied by Lemma 3.3 such that $u \leq \tilde{u}$ and $\tilde{\chi} - \chi < \delta$, where $\tilde{\chi} = \int \tilde{u}\, dm$. Denote by $\tilde{D}_\eta$ the $(\tilde{u}, \eta)$-regularity function. Then:

$$\begin{aligned} D_\varepsilon(x) &\leq \sup_{t \geq 0} \frac{\exp \int_0^t \tilde{u}(f_s x)\, ds}{e^{(\chi + \varepsilon)t}} \\ &\leq \sup_{t \geq 0} \frac{\exp \int_0^t \tilde{u}(f_s x)\, ds}{e^{(\tilde{\chi} + \varepsilon - \delta)t}} \\ &= \tilde{D}_{\varepsilon - \delta}(x), \end{aligned}$$

which lies in $L^p(m)$, for some $p > 0$, by Case 1. Therefore $D_\varepsilon \in L^p(m)$, completing the proof of Theorem A.

4. Proof of Theorem B

Let $\Phi = \{f_t\}$ be a C^2 volume preserving Anosov flow. Fix a Lyapunov bundle E corresponding to a Lyapunov exponent χ and denote the set of Lyapunov regular points by $\mathscr{R}$.

Let $x \in \mathscr{R}$ and consider the Second Variational Equation for the flow Φ on E:

$$\frac{d}{dt} T_x^E f_t = (T_{f_t x}^E X) T_x^E f_t. \tag{4.1}$$

where X is the Anosov vector field. Choose a measurable orthonormal frame $\mathbf{F} = \{F_1, \ldots, F_k\}$ for E and define a vector bundle map

$$\mathscr{T} : E \to \mathscr{R} \times \mathbb{R}^k$$

by

$$\mathscr{T}(F_i(x)) = (x, e_i),$$

where e_i is the i^{th} element of the standard basis of $\mathbb{R}^k$; extend $\mathscr{T}$ linearly over each fiber. Then $\mathscr{T}$ trivializes E, transforming (4.1) into a family of differential equations parametrized by $x \in \mathscr{R}$:

$$\dot{X} = A_x(t) X,$$

where $A_x(t)$ is the matrix of $T_{f_t x}^E X$ relative to the frame $\mathbf{F}$. As in § 2.2, for each $x \in \mathscr{R}$ we obtain an orthogonal matrix $U_x(t)$ and an upper triangular matrix $B_x(t)$ whose properties are described by Lemma 2.4. Observe that since

$$\sup_{x \in \mathscr{R}} \left\| T_x^E X \right\| \leq \sup_{x \in M} \| T_x X \| < \infty,$$

it follows that

$$\alpha = \sup\{\|A_x(t)\| : x \in \mathscr{R},\ t \in \mathbb{R}\} < \infty. \tag{4.2}$$

Furthermore, by Corollary 2.5,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t r(B_x(s))\, ds = \chi.$$

It is well-known that for every matrix M and $\delta > 0$ there exists a norm such that $\|M\| < r(M) + \delta$. The following lemma is a slight generalization of this result.

Lemma 4.1. *Let $\beta > 0$ be fixed and denote by $\mathscr{B}$ the set of all upper triangular $k \times k$ matrices such that for all $B = [b_{ij}] \in \mathscr{B}$,*

$$\max_{i<j} |b_{ij}| \leq \beta.$$

Then for every $\delta > 0$ there exists a norm $\|\cdot\|_\delta$ on $\mathbb{R}^k$ such that for all $B \in \mathscr{B}$, the induced operator matrix norm of B satisfies

$$\|B\|_\delta < r(B) + \delta.$$

Proof. The proof is an adaptation of one of the standard proofs (see, e.g., [Kre98], Theorem 3.32). Define

$$\varepsilon = \min\left\{1, \frac{\delta}{(k-1)\beta}\right\}$$

and

$$D = \operatorname{diag}(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{k-1}).$$

Then for any $B = [b_{ij}] \in \mathscr{B}$,

$$C = D^{-1}BD = \begin{bmatrix} b_{11} & \varepsilon b_{12} & \varepsilon^2 b_{13} & \cdots & \varepsilon^{k-1} b_{1k} \\ 0 & b_{22} & \varepsilon b_{23} & \cdots & \varepsilon^{k-2} b_{2k} \\ 0 & 0 & b_{33} & \cdots & \varepsilon^{k-3} b_{3k} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & b_{kk} \end{bmatrix}.$$

For a $k \times k$ matrix $A = [a_{ij}]$, write

$$\|A\|_\infty = \max_{1 \le i \le k} \sum_{j=1}^{k} |a_{ij}|.$$

Then, for all $B = [b_{ij}] \in \mathscr{B}$:

$$\|C\|_\infty \le \max_{1 \le i \le k} |b_{ii}| + (k-1)\varepsilon\beta \le r(B) + \delta.$$

We define a norm on $\mathbb{R}^k$ by

$$\|v\|_\delta = \left\|D^{-1}v\right\|_\infty,$$

where $\|(w_1, \dots, w_k)\|_\infty = \max |w_i|$. It follows that for all $B \in \mathscr{B}$,

$$\begin{aligned} \|Bv\|_\delta &= \left\|D^{-1}Bv\right\|_\infty \\ &= \left\|CD^{-1}v\right\|_\infty \\ &\le \|C\|_\infty \left\|D^{-1}v\right\|_\infty \\ &= \|C\|_\infty \|v\|_\delta \\ &\le (r(B) + \delta) \|v\|_\delta. \end{aligned}$$ □

By Lemma 2.4 and (4.2),

$$\|B_x(t)\| \le 2\|A_x(t)\| \le 2\alpha,$$

for all $x \in \mathscr{R}$ and $t \in \mathbb{R}$. Thus we can apply Lemma 4.1 to the family of matrices $\mathscr{B} = \{B_x(t) : x \in \mathscr{R}, t \in \mathbb{R}\}$. For each $\delta > 0$, this gives us a norm $\|\cdot\|_\delta$ on $\mathbb{R}^k$, which induces an operator matrix norm also denoted by $\|\cdot\|_\delta$. This yields

$$\|B_x(t)\|_\delta \le r(B_x(t)) + \delta,$$

for all $t \in \mathbb{R}$ and $x \in \mathscr{R}$.

Now consider the unique solution $X(t)$ to $\dot{X} = A_x(t)X$ satisfying the initial condition $X(0) = I$ and the corresponding solution $Z(t) = U_x(t)^{-1}X(t)$ to $\dot{Z} = B_x(t)Z$. Since $\mathbf{F}$ is orthonormal, $U_x(0) = I$, so $Z(0) = I$. Therefore,

$$Z(t) = I + \int_0^t B_x(s)Z(s)\,ds.$$

It follows that

$$\|Z(t)\|_\delta \le 1 + \int_0^t \|B_x(s)\|_\delta \|Z(s)\|_\delta\,ds,$$

so by Grönwall's inequality and Lemma 4.1,

$$\|Z(t)\|_\delta \le \exp\left\{\int_0^t \|B_x(s)\|_\delta\,ds\right\} \le e^{\delta t}\left\{\int_0^t r(B_x(s))\,ds\right\}. \tag{4.3}$$

Since $U_x(t)$ is orthogonal, its operator norm with respect to the original norm on $\mathbb{R}^k$ equals one. The old norm and the new norm on $\mathbb{R}^k$ are uniformly equivalent, so there exists a uniform constant $K_\delta > 0$ such that $\|U_x(t)\|_\delta \leq K_\delta \|U_x(t)\| = K_\delta$. Therefore,

$$\|X(t)\|_\delta = \|U_x(t)Z(t)\|_\delta \leq K_\delta \|Z(t)\|_\delta . \tag{4.4}$$

Define a function $u : \mathscr{R} \to \mathbb{R}$ by

$$u(x) = r(B_x(0)).$$

Note that in the notation of § 2.2, $u(x) = \rho_{B_x}(0)$ so by Lemma 2.7,

$$|u(x)| \leq K \|A_x(0)\| \leq K\alpha,$$

for m-a.e. x. Thus $u \in L^\infty(m)$.

We now want to show that the matrices $B_x(t)$ are well-behaved along the flow Φ.

Lemma 4.2. *It is possible to define $t \mapsto B(t)$ so that*

$$B_x(s+t) = B_{f_s x}(t),$$

for all $x \in \mathscr{R}$ and $t \in \mathbb{R}$.

Proof. Fix $x \in \mathscr{R}$. Recall that $B_x(t)$ is constructed in the following way: first, one chooses a basis $\mathbf{v} = (v_1, \ldots, v_k)$ of $\{x\} \times \mathbb{R}^k$, and applies the Gram-Schmidt procedure to the matrix $[v_1(t), \ldots, v_k(t)]$ (where $\dot{v}_i(t) = A_x(t)v_i(t)$ and $v_i(0) = v_i$), which yields a family of orthogonal matrices $U_x(t)$; then one defines $B_x(t) = U_x(t)^{-1}A_x(t)U_x(t) - U_x(t)^{-1}\dot{U}_x(t)$ (cf., 2.4). Recall also that by Lemma 2.6, $r(B_x(t))$ does not depend on the choice of $\mathbf{v}$.

Fix $s \in \mathbb{R}$. We define $B_x(t)$ and $B_{f_s x}(t)$ by choosing suitable bases $\mathbf{v} = (v_1, \ldots, v_k)$ of $\mathscr{T}(E(x)) = \{x\} \times \mathbb{R}^k$ and $\mathbf{w} = (w_1, \ldots, w_k)$ of $\mathscr{T}(E(f_s x)) = \{f_s x\} \times \mathbb{R}^k$, respectively. This is done as follows.

Define $\mathbf{v}$ by $(x, v_i) = \mathscr{T}(F_i(x))$ $(1 \leq i \leq k)$. This gives rise to a family of orthogonal matrices $U_x(t) = U_x^{\mathbf{v}}(t)$ and the corresponding family $B_x(t) = B_x^{\mathbf{v}}(t)$.

Define $\mathbf{w}$ by $(f_s x, w_i) = \mathscr{T}(T_x f_s(F_i(x)))$ $(1 \leq i \leq k)$. This gives rise to the matrices $U_{f_s x}(t) = U_{f_s x}^{\mathbf{w}}(t)$ and the corresponding family $B_{f_s x}(t) = B_{f_s x}^{\mathbf{w}}(t)$.

Let $v_i(t)$ and $w_i(t)$ be the solutions of the differential equations $\dot{v} = A_x(t)v$ and $\dot{w} = A_{f_s x}(t)w$ with initial conditions v_i and w_i, respectively. Then:

$$\begin{aligned} w_i(t) &= \mathscr{T}(T_{f_s x} f_t(T_x f_s(F_i(x)))) \\ &= \mathscr{T}(T_x f_{s+t}(F_i(x))) \\ &= v_i(s+t). \end{aligned}$$

This implies that $U_x(s+t) = U_{f_s x}(t)$.

Furthermore, since $A_x(t)$ is the matrix of $T^E_{f_t x}X$ in the frame $\mathbf{F}$, $A_x(t) = [T^E_{f_t x}X]_{\mathbf{F}}$, it follows that

$$A_x(s+t) = [T^E_{f_{s+t}x}X]_{\mathbf{F}} = [T^E_{f_t(f_s x)}X]_{\mathbf{F}} = A_{f_s x}(t).$$

Therefore,

$$\begin{aligned} B_{f_s x}(t) &= U_{f_s x}(t)^{-1}A_{f_s x}(t)U_{f_s x}(t) - U_{f_s x}(t)\dot{U}_{f_s x}(t) \\ &= U_x(s+t)^{-1}A_x(s+t)U_x(s+t) - U_x(s+t)^{-1}\dot{U}_x(s+t) \\ &= B_x(s+t). \end{aligned}$$

This completes the proof of the lemma. ∎

Note that the function u is unaffected by the choice of basis used to define $B(t)$. So for $t \mapsto B(t)$ defined as in the previous lemma, we have

$$u(f_t x) = r(B_{f_t x}(0)) = r(B_x(t)). \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), we obtain

$$\|X(t)\|_\delta \le K_\delta e^{\delta t} \exp\left\{\int_0^t u(f_s x)\, ds\right\}.$$

For each $\delta > 0$, we abuse the notation and denote the pullback of the norms $\|\cdot\|_\delta$ to E via $\mathscr{T}$ by the same symbol. That is, for each $v \in E(x)$ $(x \in \mathscr{R})$, we set

$$\|v\|_\delta = \|\mathscr{T}(v)\|_\delta .$$

This defines a family of Finsler structures on E with respect to which

$$\left\|T_x^E f_t\right\|_\delta \le K_\delta e^{\delta t} \exp\left\{\int_0^t u(f_s x)\, ds\right\},$$

for all $x \in \mathscr{R}$ and $t \ge 0$.

Since any two norms on $\mathbb{R}^k$ are uniformly equivalent, for each $\delta > 0$ there exists a constant $A_\delta > 0$ such that

$$\|v\| \le A_\delta \|v\|_\delta ,$$

for all $v \in E$, where $\|v\|$ denotes the original norm of v defined by the Riemann structure on M. It follows that the norm of $T_x^E f_t$ with respect to the original norm on E satisfies

$$\left\|T_x^E f_t\right\| \le A_\delta \left\|T_x^E f_t\right\|_\delta \le A_\delta K_\delta e^{\delta t} \exp\left\{\int_0^t u(f_s x)\, ds\right\}.$$

This completes the proof of Theorem B.

References

[Ano67] Dimitri V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proc. Steklov Math. Inst. **90** (1967), AMS Translations (1969).

[BP01] Luis Barreira and Yakov B. Pesin, *Lyapunov exponents and smooth ergodic theory*, University Lecture Series, vol. 23, American Mathematical Society, 2001.

[BR75] Rufus Bowen and David Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), no. 3, 181–202. MR 0380889 (52 #1786)

[BS00a] L. Barreira and B. Saussol, *Multifractal analysis of hyperbolic flows*, Comm. Math. Phys. **214** (2000), no. 2, 339–371. MR **1796025 (2001j:**37058)

[BS00b] Luis Barreira and J. Schmeling, *Sets of "non-typical" points have full topological entropy and full Hausdorff dimension*, Israel J. Math. **116** (2000), 29–70.

[Has94] Boris Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory Dynam. Systems (1994), 645–666.

[Has97] ——, *Regularity of the Anosov splitting II*, Ergodic Theory Dynam. Systems (1997), 169–172.

[HPS77] Morris W. Hirsch, Charles C. Pugh, and Michael Shub, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin-New York, 1977.

[Kre98] Rainer Kress, *Numerical analysis*, Grad. Text in Math., vol. 181, Springer, 1998.

[Ose68] V. I. Oseledets, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[Pet89] Karl E. Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1989.

[PS01] Ya. B. Pesin and V. Sadovskaya, *Multifractal analysis of conformal Axiom A flows*, Comm. Math. Phys. **216** (2001), no. 2, 277–312. MR **1814848 (2002g:**37035)

[Rue79] David Ruelle, *Ergodic theory of differentiable dynamical systems*, Publications Math. de l'IHES **50** (1979), 27–58.

[Wad96] Simon Waddington, *Large deviation asymptotics for Anosov flows*, Annales de l'I.H.P., section C **13** (1996), no. 4, 445–484.

[Wal99] C. P. Walkden, *Stable ergodic properties of cocycles over hyperbolic attractors*, Comm. Math. Phys. **205** (1999), no. 2, 263–281. MR **1712607 (2000f:**37037)

Department of Mathematics, San José State University, San José, CA 95192-0103
E-mail address: `simic@math.sjsu.edu`