Which Riemannian manifolds admit a geodesic flow of Anosov type?*

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In 1961 Steve Smale visited the Soviet Union and made several conjectures on the structural stability of certain toral automorphisms and geodesic flows of negative curvature. A year later D. V. Anosov proved all of Smale's conjectures; his results about geodesic flows of manifolds with strictly negative sectional curvature can be found in his paper [1].

In this paper we will describe several geometric and topological conditions that are either equivalent to or imply that the geodesic flow of a Riemannian manifold is of Anosov type. The importance of the problem stated in the title (other than its importance as a natural question to ask) is in that progress on it might contribute to the problem of which manifolds can have Riemannian metrics of negative curvature.

1 Introduction

A one-parameter family $\{\phi_t\}$ of diffeomorphisms of a manifold X is called a **flow** on X if ϕ_0 is the identity map of X and $\phi_s\phi_t = \phi_{s+t}$ for all $s, t \in \mathbf{R}$.

Let M be a complete Riemannian manifold, and let $S(M)$ denote its unit tangent bundle. If $v \in S(M)$, then there is a unique geodesic $c_v : \mathbf{R} \to M$ satisfying $c_v'(0) = v$. Recall that in a local coordinate system $\mathbf{x} = (x_1, \ldots, x_n)$, c_v satisfies the following differential equations:

$$\ddot{x}_i + \sum_{j,k} \Gamma^i_{jk}(\mathbf{x})\dot{x}_j\dot{x}_k = 0,$$

where $1 \leq i \leq n$ and Γ^i_{jk} are the Christoffel symbols. This defines the **geodesic flow** of M. Notice that the underlying manifold for this flow is $S(M)$, that is, all the geodesics are assumed to have unit speed.

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Now let $\Phi_t : E \to E$ be a one-parameter group of automorphisms of a Riemannian vector bundle E. Then Φ_t is **contracting** if there are positive constants c, λ such that

$$\|\Phi_t(v)\| \leq ce^{-\lambda t}\|v\|,$$

for all $v \in E$ and all positive t. We call Φ_t **expanding** if Φ_{-t} is contracting; this is equivalent to the existence of positive constants c', μ such that

$$\|\Phi_t(v)\| \geq c'e^{\mu t}\|v\|,$$

for all positive t and $v \in E$.

1.1 Definition An **Anosov flow** on a complete Riemannian manifold X is a flow $\{\phi_t\}$ such that the induced flow $D\phi_t : T(X) \to T(X)$ on the tangent bundle is hyperbolic in the following sense. The tangent bundle $T(X)$ can be written as the continuous Whitney sum of three invariant subbundles,

$$T(X) = E^s \bigoplus E^u \bigoplus Z,$$

where on E^s, $D\phi_t$ is contracting, on E^u, $D\phi_t$ is expanding and Z is the one-dimensional bundle defined by differentiating ϕ_t with respect to t.

So every Anosov flow defines three distributions on M. It can be shown that all of them are integrable and we denote the foliations corresponding to E^s and E^u by W^s and W^u respectively.

Now the following question becomes natural: when is the geodesic flow of M of Anosov type? The first result in this direction came from Anosov [1] who showed that the geodesic flow of negatively curved manifolds is of Anosov type and structurally stable.

To shorten the statements of theorems in the subsequent text, we will say that a manifold M whose geodesic flow is of Anosov type satisfies $(\mathcal{A})$. Also $\tilde{M}$ will always denote the universal Riemannian covering of the manifold M.

2 Topological conditions

Suppose M is a compact Riemannian manifold of dimension n satisfying $(\mathcal{A})$. Then

2.1 Theorem

(a) *The universal covering of M is diffeomorphic to $\mathbf{R}^n$.*

(b) *The fundamental group of M has exponential growth.*

(c) *Every non-trivial Abelian subgroup of the fundamental group of M is infinite cyclic (that is, isomorphic to the group* $\mathbf{Z}$*).*

(d) *All homotopy groups $\pi_k(M)$ of M vanish for $k \geq 2$.*

(e) *The geodesic flow of M is ergodic.*

(f) *The periodic orbits of the geodesic flow are dense.*

Moreover, if M satisfies $(\mathcal{A})$, and if $I : S(\tilde{M}) \to S(\tilde{M})$ is the involution defined by $I(X) = -X$, then we also have:

(g) *Each stable leaf W^s and each unstable leaf W^u different from $I(W^s)$, intersect in precisely one orbit.*

For proofs and more details, see [2].

3 Geometric conditions

In this part of the paper we will describe the geometry of a manifold satisfying $(\mathcal{A})$ in terms of various simple conditions on the Jacobi vector fields on unit speed geodesics. First we have the following

3.1 Theorem *If M is a compact Riemannian manifold satisfying $(\mathcal{A})$, then:*

(i) *There are no conjugate points on M.*

(ii) *Every closed geodesic on M has index zero.*

(iii) *The group of isometries of M is finite.*

(iv) *There are no non-trivial closed geodesics on $\tilde{M}$, and equivalently, no non-trivial closed geodesics on M which are null homotopic.*

(v) *In every connected component of the space of parametrized closed curves on M which does not contain the point curves, there is – up to the choice of initial point – precisely one closed geodesic; it has index zero.*

For a proof, see [2]. Recall that the index of a geodesic c in M is the index of the bilinear form

$$I(X,Y) = -\int_0^L g(X, Y'' + R(c', Y)c')ds,$$

where X and Y are vector fields along c vanishing at 0 and L is the length of c.

Now assume M is a complete, C^∞ Riemannian manifold of dimension $n \geq 2$ without conjugate points. Also assume that the universal Riemannian cover $\tilde{M}$ of M is compactly homogeneous. (A Riemannian manifold X is said to be **compactly homogeneous** if there exists a compact set K in X such that X is the union of translates of K by isometries of X. For example, if X is homogeneous or admits a compact Riemannian quotient, then X is compactly homogeneous.) Then we have

3.2 Theorem *The following are equivalent:*

1) *M satisfies $(\mathcal{A})$.*

2) *There exists no nonzero perpendicular Jacobi field Y on a unit speed geodesic of M such that $\|Y(t)\|$ is bounded above for all real t.*

3) *The following two conditions hold:*

(*i*) *There exist numbers $a > 0$, $s_0 \geq 0$ such that for every perpendicular Jacobi field Y on a unit speed geodesic c with $Y(0) = 0$ we have*

$$\|Y(t)\| \geq a\|Y(s)\|,$$

for every $t \geq s \geq s_0$.

(*ii*)

$$\int_1^\infty \frac{dt}{\psi(t)} < \infty,$$

where $\psi(t) = \inf\{\|Y(t)\| : \ Y(0) = 0, \ \|Y'(0)\| = 1, Y \in \mathcal{J}\}$ and $\mathcal{J}$ is the space of all Jacobi fields perpendicular to some maximal geodesic of M.

Let c and σ be two maximal geodesics such that $c'(0)$ has unit length and is perpendicular to $\sigma'(0)$. A Jacobi vector field Y is called a σ-Jacobi field if $Y(0) \neq 0$, $Y(0)$ is tangent to σ, Y is perpendicular to c, and $\langle Y(0), Y'(0)\rangle = 0$.

3.3 Definition We say that the point $c(\tau)$ is a **focal point** of σ along c if there exists a nontrivial σ-Jacobi field Y along c such that $Y(\tau) = 0$.

More generally, a point $q \in M$ is a focal point of a submanifold N of M if $q = \exp_p(v)$ for some $p \in M$ and some v in the orthogonal complement of T_pN in T_pM, and $T_v \exp_p$ is singular. We say that M has **no focal points** if no maximal geodesic σ has focal points along any unit speed geodesic perpendicular to σ.

The "no focal point" property is equivalent to the following. Let c be a unit speed geodesic of M and let Y be a (not necessarily perpendicular) Jacobi vector field on c such that $Y(0) = 0$ and $Y'(0) \neq 0$. Then for every $t > 0$:

$$\frac{d}{dt}\|Y(t)\|^2 > 0.$$

Thus, if M has no focal points, then it also has no conjugate points. For example, if the sectional curvature of M is nonpositive, then M has no focal points.

3.4 Corollary *Suppose M has no focal points. Then the following are equivalent:*

1) *M satisfies ($\mathcal{A}$).*

2) *There exists no nonzero perpendicular* parallel *Jacobi field Y on a unit speed geodesic of M.*

3) $\int_1^\infty \frac{dt}{\psi(t)} < \infty$.

3.5 Corollary *Suppose that M satisfies ($\mathcal{A}$). If c is an arbitrary unit speed geodesic of M and $E(t)$ a nonzero perpendicular parallel vector field on c, then there exists a real number t such that the sectional curvature*

$$K(E(t), c'(t)) < 0.$$

3.6 Corollary *If M has no focal points and M satisfies the property of the previous Corollary, then M satisfies ($\mathcal{A}$).*

3.7 Corollary *If M is two-dimensional and without focal points, then ($\mathcal{A}$) is equivalent to: every geodesic passes through at least one point of negative Gaussian curvature.*

Proofs of these statements can be found in [3].

Now suppose that M has no focal points. Then the universal covering $\tilde{M}$ of M also has no focal points and satisfies the following property: let c be a maximal geodesic of $\tilde{M}$ and p a point in $\tilde{M}$ not lying in the image of c; then there exists a unique geodesic from p perpendicular to c. If q is the point on c closest to p, then the unique geodesic from p to q is perpendicular to c. This defines a map

$$P_c : \tilde{M} \to c,$$

which sends each point p of $\tilde{M}$ to the point q on c nearest to p. This map is of class C^∞.

3.8 Theorem *Let M be a complete Riemannian manifold of dimension $n \geq 2$, without focal points, and suppose $\tilde{M}$ of M is compactly homogeneous. Then the following conditions are equivalent and imply the condition ($\mathcal{A}$). Furthermore, if M has nonpositive sectional curvature, then these conditions are equivalent to ($\mathcal{A}$).*

(a) *There exists a positive constant τ such that for every maximal geodesic* *c in* $\tilde{M}$ *and for every nonzero vector* $v \in T_p\tilde{M}$ *with* $\mathrm{dist}(p, c) \geq \tau$*, we have*

$$\|T_pP_c(v)\| < \|v\|.$$

(b) *There exist positive constants* α, λ *such that for every maximal geodesic c in* $\tilde{M}$ *and every vector* $v \in T_p\tilde{M}$*,*

$$\|T_pP_c(v)\| \leq \alpha\|v\| \exp[-\lambda \ \mathrm{dist}(p, c)].$$

(c) *There exists a positive number* τ *such that if* Y *is a perpendicular Jacobi field on a unit speed geodesic* c *of* M*,* $Y(0) \neq 0$*, and* $\langle Y(0), Y'(0)\rangle = 0$*, then*

$$\|Y(t)\| > \|Y(0)\|$$

for all $t > \tau$*.*

(d) *There exists a point* $p \in M$ *and positive constants* β, τ *such that if* Y *is a perpendicular Jacobi field on a unit speed geodesic* c *of* M *with* $c(0) = p, Y(0) = 0$*, and* $Y'(0) \neq 0$*, then*

$$\frac{d}{dt} \log \|Y(t)\|^2 \geq \beta,$$

for all $t \geq \tau$*.*

(e) *There exist positive constants* β, τ *such that if* Y *is a perpendicular Jacobi field on a unit speed geodesic* c *of* M*, with* $Y(0) = 0, Y'(0) \neq 0$*, then*

$$\frac{d}{dt} \log \|Y(t)\|^2 \geq \beta,$$

for all $t \geq \tau$*.*

(f) *There exist positive constants* β, τ *such that for every perpendicular Jacobi vector field* Z *on a unit speed geodesic* c *of* M *with* $\langle Z(0), Z'(0)\rangle \geq 0$*,*

$$\frac{d}{dt} \log \|Z(t)\|^2 \geq \beta,$$

for all $t \geq \tau$*.*

The constants appearing in different parts of this theorem are not necessarily the same. Theorem 3.8 was proven in [3].

4 Conclusion

As remarked above, if a manifold has negative sectional curvature, then its geodesic flow satisfies ($\mathcal{A}$). The corollaries in section 3 indicate (at least intuitively) that if a manifold satisfies ($\mathcal{A}$), then it must admit a metric of nonpositive sectional curvature. It is then natural to ask: to what extent can the sectional curvature of such a manifold be positive? In [3] P. Eberlein constructed an example of a compact Riemannian manifold that satisfies ($\mathcal{A}$) and admits a metric with nonpositive sectional curvature and sectional curvature identically zero on large open sets.

But it remains an open question to prove or disprove that if a manifold satisfies ($\mathcal{A}$), then it admits a metric of nonpositive sectional curvature.

References

[1] D.V. Anosov: Geodesic flows on closed Riemannian manifolds with negative curvature, *Proceedings of Steklov Math. Institute* **90** (1967), pp. 1-235, AMS, Providence, RI

[2] W. Klingenberg: Riemannian manifolds with geodesic flow of Anosov type, *Annals of Mathematics* (2) **99** (1974), pp. 1-13

[3] P. Eberlein: When is a geodesic flow of Anosov type? I, II, *Journal of Differential Geometry* **8** (1973), pp. 437-463, ibid. **8** (1973), pp. 565-577

[4] S. Smale: Differentiable dynamical systems, from *The Mathematics of Time*, pp. 1-82, Springer-Verlag, New York 1980.

[5] S. Gallot, D. Hulin, J. Lafontaine: *Riemannian Geometry*, 2^{nd} edition, Universitext, Springer-Verlag, New York 1990