

# A LOWER BOUND ON THE SUBRIEMANNIAN DISTANCE FOR HÖLDER DISTRIBUTIONS

SLOBODAN N. SIMIĆ

ABSTRACT. Whereas subriemannian geometry usually deals with smooth horizontal distributions, partially hyperbolic dynamical systems provide many examples of subriemannian geometries defined by non-smooth (namely, Hölder continuous) distributions. These distributions are of great significance for the behavior of the parent dynamical system. The study of Hölder subriemannian geometries could therefore offer new insights into both dynamics and subriemannian geometry. In this paper we make a small step in that direction: we prove a Hölder-type lower bound on the subriemannian distance for Hölder continuous codimension one horizontal distributions. This bound generalizes the well-known square root bound valid in the smooth case.

## 1. INTRODUCTION

The purpose of this note is to prove a lower bound in terms of the Riemannian distance for subriemannian geometries defined by a codimension one distributions that are only Hölder continuous, generalizing the well-known square root bound valid in the smooth case.

A subriemannian or Carnot-Carathéodory geometry is a pair  $(M, H)$ , where  $M$  is a smooth manifold  $M$  and  $H$  is a nowhere integrable distribution endowed with a Riemannian metric  $g$ . We call  $H$  nowhere integrable on  $M$  if every two points  $p, q \in M$  can be connected by a piecewise  $C^1$  path almost everywhere tangent to  $H$ . Classically, both  $H$  and  $g$  are assumed to be smooth, e.g., of class  $C^\infty$ ;  $H$  is usually called a horizontal distribution (or a polarization, by Gromov [Gro96]) and paths a.e. tangent to it, horizontal paths. The subriemannian distance is defined by

$$d_H(p, q) = \inf\{|\gamma| : \gamma \text{ is a horizontal path from } p \text{ to } q\},$$

where the length  $|\gamma|$  of a horizontal path  $\gamma : I \rightarrow M$  defined in the usual way,  $|\gamma| = \int_I g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt$ .

By the well-known theorem of Chow and Rashevskii [Mon02], bracket generating smooth distributions are nowhere integrable, where  $H$  is called bracket generating if any local smooth frame  $\{X_1, \dots, X_k\}$  for  $H$ , together with all of its iterated Lie brackets spans the whole tangent bundle of  $M$ . (In PDEs, the bracket-generating condition is also called the Hörmander condition.) However,  $H$  does not have to be bracket generating, or even

---

*Date:* June 29, 2009.

smooth, to define a subriemannian geometry. The most important examples of such distributions come from dynamical systems. Let  $f : M \rightarrow M$  be a  $C^r$  ( $r \geq 1$ ) partially hyperbolic diffeomorphism of a smooth compact Riemannian manifold  $M$ . This means that  $f$  is like an Anosov diffeomorphism, but in addition to hyperbolic behavior (i.e., exponential contraction and exponential expansion),  $f$  also exhibits non-hyperbolic behavior along certain tangent directions. More precisely,  $f$  is partially hyperbolic if the tangent bundle of  $M$  splits continuously and invariantly into the stable, center, and unstable bundle (i.e., distribution),

$$TM = E^s \oplus E^c \oplus E^u,$$

such that the tangent map  $Tf$  of  $f$  exponentially contracts  $E^s$ , exponentially expands  $E^u$  and this hyperbolic action on  $E^s \oplus E^u$  dominates the action of  $Tf$  on  $E^c$  (see [Pes04, PS04]). The stable and unstable bundle are always uniquely integrable, giving rise to the stable and unstable foliations,  $W^s, W^u$ .

A partially hyperbolic diffeomorphism is (rather unfortunately) called *accessible* if every two points of  $M$  can be joined by an *su*-path, i.e., a continuous piecewise smooth path consisting of finitely many arcs lying in a single leaf of  $W^s$  or a single leaf of  $W^u$ . If  $f$  is accessible, then the distribution  $E^s \oplus E^u$  is clearly nowhere integrable. Dolgopyat and Wilkinson [DW03] proved that in the space of  $C^r$  ( $r \geq 1$ ) partially hyperbolic diffeomorphisms there is a  $C^1$  open and dense set of accessible diffeomorphisms. Since invariant distributions of a partially hyperbolic diffeomorphism are in general only  $C^\theta$ , for some  $\theta \in (0, 1)$  (see [HPS77, Pes04]), this result shows that there is an abundance of non-smooth, Hölder continuous nowhere integrable distributions. Since these distributions play a crucial role in the dynamics of partially hyperbolic systems and the accessibility property is frequently key to ergodicity (cf., e.g., [PS04]), it is clear that a better understanding of subriemannian geometries defined by Hölder distributions would be very useful in dynamical systems. This paper provides a small initial step in that direction.

For smooth horizontal distributions  $H$ , the relation between the subriemannian distance  $d_H$  and the Riemannian distance, which we denote by  $\text{dist}$ , is well understood and is characterized by the Ball-Box theorem [Mon02, Gro96]. When  $H$  is of codimension one, this theorem states that in the vertical direction, i.e., along any short smooth path  $\gamma$  transverse to  $H$ ,  $d_H$  is equivalent to  $\sqrt{\text{dist}}$ . That is, there exist constants  $a, b > 0$  such that for all  $p, q$  on  $\gamma$ ,

$$a\sqrt{\text{dist}(p, q)} \leq d_H(p, q) \leq b\sqrt{\text{dist}(p, q)}. \quad (1)$$

In the horizontal direction, i.e., along any horizontal path,  $d_H$  and  $\text{dist}$  are clearly equivalent. This means that subriemannian geometry is non-isotropic: it behaves differently in different directions. Subriemannian spheres are far from being “round”. In the Heisenberg group, for instance, subriemannian spheres look like an apple [Mon02].

In [Gro96], M. Gromov gave a short and elegant proof (without details) of the lower bound for  $d_H$ , i.e., the left hand side of (1). His proof uses the assumption that  $H$  is  $C^1$  only in the following way: if  $\alpha$  is a 1-form such that  $\text{Ker}(\alpha) = H$ , then

$$\left| \int_{\partial D} \alpha \right| \leq K |D|, \quad (2)$$

for every  $C^1$  immersed 2-disk with piecewise  $C^1$  boundary, where  $K$  is a constant independent of  $D$ . Here,  $|D|$  denotes the area of  $D$ . If  $H$  (hence  $\alpha$ ) is  $C^1$ , this follows directly from the Stokes theorem.

Gromov also remarks that without the  $C^1$  assumption, the square root estimate probably fails. We will show that this is indeed the case, in the sense that if  $H$  is only  $C^\theta$ , for some  $0 < \theta < 1$ , then in the vertical direction,  $d_H(p, q) \geq C \text{dist}(p, q)^{1/(1+\theta)}$ . To generalize Gromov's approach to  $C^\theta$  horizontal distributions, one needs a generalization of the estimate (2) to forms  $\alpha$  that are only Hölder. One such estimate was recently proved in [Sim09] and states that the integral of a  $C^\theta$   $k$ -form  $\alpha$  over the boundary of a sufficiently small  $(k+1)$ -disk  $D$  is bounded by a certain multiplicative convex combination of the  $(k+1)$ -volume  $|D|$  of  $D$  and the  $k$ -dimensional area  $|\partial D|$  of its boundary; see Theorem 2.1 in the next section.

Our main result is the following:

**Theorem.** *Suppose that  $H$  is a nowhere integrable codimension one distribution of class  $C^\theta$ , for some  $0 < \theta < 1$ , on a smooth compact Riemannian manifold  $M$ . Assume that the Riemannian metric on  $H$  is the restriction of the ambient metric from  $M$ . Then there exists a constant  $\varrho > 0$  such that for any  $C^1$  path  $\gamma$  transverse to  $H$  and for every two points  $p, q$  on  $\gamma$  with Riemannian distance less than  $\varrho$ , we have*

$$d_H(p, q) \geq C \text{dist}(p, q)^{\frac{1}{1+\theta}}, \quad (3)$$

where  $C > 0$  is a constant that depends only on  $H$  and  $\gamma$ .

- Remark.**
- (a) Observe that if  $H$  is  $C^1$ , then it is  $C^\theta$  for all  $0 < \theta < 1$ , so we obtain the old quadratic estimate by letting  $\theta \rightarrow 1-$ .
  - (b) Also note that if  $\text{dist}(p, q) = \varepsilon$  is small, then  $\varepsilon^{1/2} \gg \varepsilon^{1/(1+\theta)}$ , which means that the lower bound on  $d_H(p, q)$  is tighter for  $C^1$  distributions than for  $C^\theta$  ones.
  - (c) It is clear that in any horizontal direction,  $d_H$  is equivalent to  $\text{dist}$ .

## 2. AUXILIARY RESULTS

The main tool in the proof will be the following inequality.

**2.1. Theorem** (Theorem A, [Sim09]). *Let  $M$  be a compact manifold and let  $\alpha$  be a  $C^\theta$   $k$ -form on  $M$ , for some  $0 < \theta < 1$  and  $1 \leq k \leq n-1$ . There exist constants  $\sigma, K > 0$ , depending only on  $M$ ,  $\theta$ , and  $k$ , such that for every*

$C^1$ -immersed  $(k+1)$ -disk  $D$  in  $M$  with piecewise  $C^1$  boundary satisfying  $\max\{\text{diam}(\partial D), |\partial D|\} < \sigma$ , we have

$$\left| \int_{\partial D} \alpha \right| \leq K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta.$$

As before,  $|D|$  denotes the  $(k+1)$ -volume of  $D$  and  $|\partial D|$ , the  $k$ -volume (or area) of its boundary.

**Remark.** If  $k = 1$ , then  $\text{diam}(\partial D) \leq |\partial D|$ , so the assumption  $\text{diam}(\partial D) < \sigma$  is superfluous.

The Hölder norm of  $\alpha$  on  $M$  is defined in a natural way as follows. Let  $\mathcal{A} = \{(U, \varphi)\}$  be a finite  $C^\infty$  atlas of  $M$ . We say that  $\alpha$  is  $C^\theta$  on  $M$  if  $\alpha$  is  $C^\theta$  in each chart  $(U, \varphi)$ ; i.e.,  $(\varphi^{-1})^* \alpha$  is  $C^\theta$ , for each  $(U, \varphi) \in \mathcal{A}$ . We set

$$\|\alpha\|_{C^\theta} = \max_{(U, \varphi) \in \mathcal{A}} \|(\varphi^{-1})^* \alpha\|_{C^\theta(\varphi(U))}.$$

For a  $C^\theta$  form  $\alpha = \sum a_I dx_I$  defined on an open subset  $U \subset \mathbb{R}^n$ , we set

$$\|\alpha\|_{C^\theta(U)} = \max_I \|a_I\|_{C^\theta},$$

and for a bounded function  $f : U \rightarrow \mathbb{R}$ ,

$$\|f\|_{C^\theta(U)} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\theta}.$$

We will also need the following version of the solution to the isoperimetric problem “in the small”.

**2.2. Lemma** ([Gro83], **Sublemma 3.4.B'**). *For every compact manifold  $M$ , there exists a small positive constant  $\delta_M$  such that every  $k$ -dimensional cycle  $Z$  in  $M$  of volume less than  $\delta_M$  bounds a chain  $Y$  in  $M$ , which is small in the following sense:*

- (i)  $|Y| \leq c_M |Z|^{(k+1)/k}$ , for some constant  $c_M$  depending only on  $M$ ;
- (ii) The chain  $Y$  is contained in the  $\varrho$ -neighborhood of  $Z$ , where  $\varrho \leq c_M |Z|^{1/k}$ .

The following corollary is immediate.

**2.3. Corollary.** *If  $\Gamma$  is a closed piecewise  $C^1$  path in a compact manifold  $M$  with  $|\Gamma| < \delta_M$ , then there exists a 2-disk  $D \subset M$  such that  $\partial D = \Gamma$ ,*

$$|D| \leq c_M |\Gamma|^2, \tag{4}$$

*and  $D$  is contained in the  $\varrho$ -neighborhood of  $\Gamma$ , where  $\varrho \leq c_M |\Gamma|$ .*

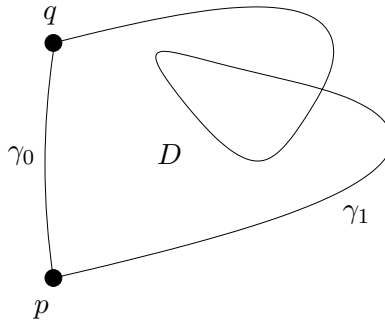


FIGURE 1. Vertical path  $\gamma_0$  and horizontal path  $\gamma_1$ .

### 3. PROOF OF THE THEOREM

We follow Gromov's proof in [Gro96], p. 116. Since the statement is local and concerns arbitrary directions transverse to  $H$ , we can assume without loss of generality that  $\gamma$  is a Riemannian geodesic.

Define a 1-form  $\alpha$  on  $M$  by

$$\text{Ker}(\alpha) = H, \quad \alpha(\dot{\gamma}) = 1.$$

Let  $\tau = \min(\delta_M, \sigma)$ , where  $\delta_M, \sigma$  are the constants from Theorem 2.1 and Lemma 2.2.

Since  $d_H : M \times M \rightarrow [0, \infty)$  is continuous with respect to  $\text{dist}$ , and  $M \times M$  is compact, it follows that  $d_H$  is uniformly continuous relative to  $\text{dist}$ . Therefore, there exists  $\eta > 0$  such that for all  $x, y \in M$ , if  $\text{dist}(x, y) < \eta$ , then have  $d_H(x, y) < \tau/2$ .

Set

$$\varrho = \min \left( \eta, \frac{\tau}{2}, \frac{1}{2^{\frac{1+\theta}{\theta}} c_M K^{1/\theta} \|\alpha\|_{C^\theta}^{1/\theta}} \right).$$

Observe that if  $\text{dist}(x, y) < \varrho$ , then  $\text{dist}(x, y) + d_H(x, y) < \tau$ .

Take any two points  $p, q$  on  $\gamma$  satisfying  $\text{dist}(p, q) < \varrho$ . Denote the segment of  $\gamma$  starting at  $p$  and ending at  $q$  by  $\gamma_0$ . Let  $\varepsilon > 0$  be arbitrary but sufficiently small so that  $\text{dist}(p, q) + d_H(p, q) + \varepsilon < \delta_M$ . Finally, let  $\gamma_1$  be a horizontal path from  $p$  to  $q$  with  $|\gamma_1| < d_H(p, q) + \varepsilon$ . (Note that a subriemannian geodesic from  $p$  to  $q$  may not exist, since  $H$  is assumed to be only Hölder.) Define  $\Gamma = \gamma_0 - \gamma_1$ ;  $\Gamma$  is a closed piecewise  $C^1$  path.

Since

$$|\Gamma| = |\gamma_0| + |\gamma_1| < \text{dist}(p, q) + d_H(p, q) + \varepsilon < \tau < \delta_M,$$

by Corollary 2.3 there exists a 2-disk  $D$  such that  $\partial D = \Gamma$  and  $|D| \leq c_M |\Gamma|^2$  (see Fig. 1). On the other hand, since  $\tau < \sigma$ , it follows that  $|\partial D| = |\Gamma| < \sigma$ .

Therefore, we can apply Theorem 2.1 to  $\alpha$  on  $D$ . This yields:

$$\begin{aligned}
|\gamma_0| &= \int_{\partial D} \alpha \\
&\leq K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} |D|^\theta \\
&\leq K \|\alpha\|_{C^\theta} |\partial D|^{1-\theta} (c_M |\partial D|^2)^\theta \\
&= c_M^\theta K \|\alpha\|_{C^\theta} |\partial D|^{1+\theta} \\
&= c_M^\theta K \|\alpha\|_{C^\theta} (|\gamma_0| + |\gamma_1|)^{1+\theta} \\
&< c_M^\theta K \|\alpha\|_{C^\theta} \{|\gamma_0| + d_H(p, q) + \varepsilon\}^{1+\theta}.
\end{aligned}$$

Regrouping terms, we obtain

$$|\gamma_0|^{\frac{1}{1+\theta}} - c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} |\gamma_0| < c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} \{d_H(p, q) + \varepsilon\}.$$

Factoring  $|\gamma_0|^{\frac{1}{1+\theta}}$  out, the left-hand side becomes

$$|\gamma_0|^{\frac{1}{1+\theta}} \left\{ 1 - c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} |\gamma_0|^{\frac{\theta}{1+\theta}} \right\}.$$

It is not hard to see that the assumption  $|\gamma_0| = \text{dist}(p, q) < \varrho$  implies that the quantity in the curly braces is  $\geq 1/2$ . Therefore,

$$\frac{1}{2} |\gamma_0|^{\frac{1}{1+\theta}} < c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} \{d_H(p, q) + \varepsilon\}.$$

Since  $\varepsilon > 0$  can be arbitrarily small and  $|\gamma_0| = \text{dist}(p, q)$ , we obtain

$$\text{dist}(p, q)^{\frac{1}{1+\theta}} \leq 2c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} d_H(p, q).$$

This completes the proof with

$$C = \left( 2c_M^{\frac{\theta}{1+\theta}} K^{\frac{1}{1+\theta}} \|\alpha\|_{C^\theta}^{\frac{1}{1+\theta}} \right)^{-1}. \quad \square$$

The following question appears to be much harder:

**Question.** *Is there an analogous upper bound for  $d_H$  with respect to  $\text{dist}$  when  $H$  is  $C^\theta$ ?*

This and essentially all other questions of Hölder subriemannian geometry remain wide open.

## REFERENCES

- [DW03] D. Dolgopyat and A. Wilkinson, *Stable accessibility is  $C^1$  dense*, *Astérisque* **xvii** (2003), no. 287, 33–60.
- [Gro83] Mikhail Gromov, *Filling Riemannian manifolds*, *J. Diff. Geometry* **18** (1983), 1–147.
- [Gro96] ———, *Carnot-Carathéodory spaces seen from within*, *Sub-Riemannian Geometry* (A. Bellaïche and J.-J. Risler, eds.), *Progress in Mathematics*, vol. 144, Birkhäuser, 1996, pp. 79–323.
- [HPS77] Morris W. Hirsch, Charles C. Pugh, and Michael Shub, *Invariant manifolds*, *Lecture Notes in Mathematics*, vol. 583, Springer-Verlag, Berlin-New York, 1977.

- [Mon02] Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, AMS, 2002.
- [Pes04] Yakov B. Pesin, *Lectures on partial hyperbolicity and stable ergodicity*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2004.
- [PS04] Charles C. Pugh and Michael Shub, *Stable ergodicity*, Bull. Amer. Math. Soc. **41** (2004), 1–41.
- [Sim09] Slobodan N. Simić, *Hölder forms and integrability of invariant distributions*, Discrete Contin. Dyn. Syst. **25** (2009), no. 2, 669–685.

DEPARTMENT OF MATHEMATICS, SAN JOSÉ STATE UNIVERSITY, SAN JOSÉ, CA 95192-0103

*E-mail address:* `simic@math.sjsu.edu`