VOLUME PRESERVING CODIMENSION ONE ANOSOV FLOWS IN DIMENSIONS GREATER THAN THREE ARE SUSPENSIONS

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Abstract. We show that every volume preserving codimension one Anosov flow on a closed Riemannian manifold of dimension greater than three admits a global cross section and is therefore topologically equivalent to a suspension of a linear toral automorphism. This proves a conjecture of Verjovsky from the 1970's in the volume preserving case.

1. Introduction

The theory of hyperbolic dynamical systems, despite its long history, still abounds with open fundamental problems. Among these is the following

Conjecture. *Every codimension one Anosov flow on a closed Riemannian manifold of dimension greater than three admits a global cross section.*

Verjovsky stated the conjecture in [34] for all dimensions with an additional assumption that the fundamental group of the manifold is solvable. This was proved by Plante [21, 22] and Armendariz [4], who showed that the conjecture is true if and only if the fundamental group of the manifold is solvable. In the above form, the conjecture first appeared in Ghys [13]. However, Ghys has pointed out that Verjovsky had originally proposed it in the 1970's. In [13], Ghys showed that the conjecture is true if the sum $E^{su} = E^{ss} \oplus E^{uu}$ of the strong bundles of the flow is of class C^1 or if the codimension one center stable bundle E^{cs} is C^2 and the flow preserves volume. The first result of Ghys was generalized in [29] to Lipschitz E^{su}. The second one was extended in [30] to the case when E^{su} is Lip− or when E^{cs} is $C^{1+\text{Lip}-}$, where Lip− means C^θ for all $\theta \in (0,1)$. In a related work, Bonatti and Guelman [8] showed that if the time one map of a codimension one Anosov flow can be C^1 approximated by an Axiom A diffeomorphism with more than one attractor, then the flow is topologically equivalent to the suspension of an Anosov diffeomorphism.

In this paper, we prove the following result.

Main Theorem. *Verjovsky's conjecture is true for volume preserving flows. More precisely, every volume preserving codimension one Anosov flow on a closed Riemannian manifold of dimension greater than three can be C^1 approximated by a C^∞ flow of the same type whose synchronization admits a global cross section with constant first-return time.*

By synchronization we mean a suitable reparametrization of the flow that makes the strong unstable cocycle be of the form e^{ct}, i.e., independent of the space variable (see §3-B). The induced Poincaré map on a global cross section is automatically a codimension one Anosov diffeomorphism, f. Franks [11] proved that if the non-wandering set of f is the whole manifold, then f is topologically conjugate to a linear toral automorphism. By a result of Newhouse [17], this is indeed the case for every codimension one Anosov diffeomorphism. We therefore obtain the following classification.

Classification Theorem. *Every volume preserving codimension one Anosov flow on a closed Riemannian manifold of dimension greater than three is topologically equivalent to a suspension of a linear toral automorphism.*

Key words and phrases. Anosov flow, cross section, suspension, Lyapunov exponent.

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Recall that two flows are topologically equivalent if there exists a homeomorphism which takes orbits of one flow to orbits of the other, preserving the orientation but not necessarily preserving the time parameter.

Outline of the proof. Given a C^1 volume preserving codimension one Anosov flow on a C^∞ closed Riemannian manifold M of dimension $n > 3$, the goal is to show there exists a topologically equivalent flow with jointly integrable (see §2) strong foliations. The main difficulty is the lack of smoothness of the strong stable distribution E^{ss}.

We start by C^1 approximating the original flow by a C^1 flow with a continuous Oseledets splitting (Step 1). To do this, we use the work of Bochi-Viana [7] and Bessa [6] (see §3-A). Next, we use the density result of Arbieto and Matheus [3] to C^1 approximate again. We obtain a C^∞ volume preserving codimension one Anosov flow such that either: (A) the dimension of the top Lyapunov bundle is one and the sum of the remaining Lyapunov bundles in the strong stable bundle is continuous on the whole manifold, or (B) the top Lyapunov exponent of its synchronization is less than $\tau = (2-\theta)^{-1}$, where θ is the Hölder exponent of its strong stable bundle.

Next, we *synchronize* (§3-B) the flow to obtain a $C^{1+\text{Hölder}}$ volume preserving codimension one Anosov flow $\{f_t\}$, topologically equivalent to the original one, and satisfying $\det Tf_t \restriction_{E^{uu}} \equiv e^t$ (Step 2). For the reverse flow f_{-t}, the Oseledets splitting $TM = E_1 \oplus \cdots \oplus E_\ell$, corresponding to Lyapunov exponents $\chi_1 < \cdots < \chi_\ell$, satisfies either (A) or (B) above. Observe that $E_1 = E^{uu}$, $E_2 = E^c$, and $E_3 \oplus \cdots \oplus E_\ell = E^{ss}$. We show that for this flow, the foliations W^{ss} and W^{uu} are jointly integrable.

For any $p \in M$ and $q \in W^{ss}_{\text{loc}}(p)$, let $\mathbf{h}^{uu}_{p,q} : W^{cs}_{\text{loc}}(p) \to W^{cs}_{\text{loc}}(q)$ be the strong unstable holonomy (§3-C). We prove that $T\mathbf{h}^{uu}_{p,q}$ takes E^{ss} to itself. In case (A), this is done in Steps 3A, 4, and 5. In case (B), it is done in Steps 3B and 5.

In Step 3A, we show $T\mathbf{h}^{uu}_{p,q}(F_{\ell-1}) \subset E^{ss}$, where $F_{\ell-1} = E_3 \oplus \cdots \oplus E_{\ell-1}$ is the invariant subbundle of E^{ss} consisting of vectors whose growth rate relative to Tf_{-t} is not the maximal possible one, χ_ℓ. In Step 3B, which treats case (B), we show that $T\mathbf{h}^{uu}_{p,q}$ takes the *whole* bundle E^{ss} onto itself.

Step 4, which is a continuation of Step 3A, shows that $T\mathbf{h}^{uu}_{p,q}$, in fact, takes the whole bundle E^{ss} onto itself. This is done using some simple linear algebra of differential forms.

Step 5 completes the proof by showing how $T\mathbf{h}^{uu}_{p,q}(E^{ss}) = E^{ss}$ implies the existence of a global cross section to the original flow.

The proofs of Steps 3A and 3B are based on one key estimate (Theorem 3.13). Let α be a 1-form on M dual to $E^{su} = E^{ss} \oplus E^{uu}$, defined by

$$\operatorname{Ker}(\alpha) = E^{su} \qquad \text{and} \qquad \alpha(X) = 1. \tag{1.1}$$

By Proposition 3.7, joint integrability of W^{ss} and W^{uu} is equivalent to the vanishing of the integral of α over the boundary of any (small) *su*-disk D. An *su*-disk D (§3-C) is a smooth 2-disks foliated by arcs of strong unstable manifolds, with piecewise smooth boundary consisting of two opposing strong unstable arcs, and one strong stable arc (the "base" of D) opposite a center stable arc (FIG. 1). If D is an *su*-disk, then (Proposition 3.10) the area of $f_{-t}D$ tends to zero, as $t \to \infty$. To take advantage of this fact we need a suitable estimate of the integral of α over ∂D which involves the area of D. If α were C^1, then by Stokes' theorem, $\left|\int_{\partial D} \alpha\right| \leq \|\alpha\|_{C^1} |D|$. For general Hölder forms, such estimates are hard to come by and are not suitable for our purposes. However, for the very special form α, it is possible to derive an estimate in terms of both the circumference $|\partial D|$ and area $|D|$. The derivation is based on an analysis trick, which goes as follows. We regularize α to obtain a smooth form α^ε such that $\|\alpha^\varepsilon - \alpha\|_{C^0} \lesssim \varepsilon^\theta$. However, along W^{cs}-leaves, we can ensure $\|(\alpha^\varepsilon - \alpha)\restriction_{W^{cs}}\|_{C^0} \lesssim \varepsilon$. This yields, essentially, $\left|\int_{\partial D} \alpha\right| \lesssim |\partial D|\,\varepsilon + |D|\,\varepsilon^{\theta-1}$. The trick is to minimize over ε. If ε is allowed to range over a sufficiently large interval $(0, \varepsilon_0)$, then the minimum of the right hand side is $|\partial D|^{1-\tau}\,|D|^\tau$, where $\tau = 1/(2-\theta)$.

To show that the integral of α over ∂D vanishes, we use the flow invariance of α, $f_t^*\alpha = \alpha$, and apply the key estimate to the integral of α over $\partial f_{-t}D$. For this to work, we need to decompose

D into smaller su-disks (FIG. 2). We obtain $\left|\int_{\partial f_{-t}D} \alpha\right| \lesssim |\partial f_{-t}D|^{1-\tau}\ |f_{-t}D|^{\tau}$ (cf., (4.5)). In case (A), the right hand side goes to zero, as $t \to \infty$, if the base γ of D lies in a certain open set of full measure and is tangent to $F_{\ell-1}$, i.e., if the length of $f_{-t}(\gamma)$ does not grow at the fastest possible speed, $e^{\chi_\ell t}$. This implies that $T\mathbf{h}^{uu}_{p,q}(F_{\ell-1}) \subset E^{ss}$. In case (B), the same statement holds for all su-disks D in an open set of full measure, implying $T\mathbf{h}^{uu}_{p,q}(E^{ss}) \subset E^{ss}$.

The paper is organized as follows. In §2, we review the necessary basics of Anosov flows and the existence of global cross sections. In §3, we prove a series of preparatory results on Lyapunov exponents (§3-A), synchronization (§3-B), su-disks (§3-C), and regularization (§3-D). The key estimate is proved in §3-E. The proof of the main theorem is given in §4.

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2. ANOSOV FLOWS, CROSS SECTIONS, AND SUSPENSIONS

A non-singular smooth flow $\Phi = \{f_t\}$ on a closed (compact and without boundary) Riemannian manifold M is called Anosov if there exists a Tf_t-invariant continuous splitting of the tangent bundle,

$$TM = E^{uu} \oplus E^{c} \oplus E^{ss},$$

and constants $C > 0$, $0 < \nu < 1$, and $\lambda > 1$ such that for all $t \geq 0$,

$$\|Tf_t\restriction_{E^{ss}}\| \leq C\nu^t \qquad \text{and} \qquad \|Tf_t\restriction_{E^{uu}}\| \geq C\lambda^t.$$

The center bundle E^c is one dimensional and generated by the vector field X tangent to the flow.

We call $E^{uu}, E^{ss}, E^{cu} = E^c \oplus E^{uu}$, and $E^{cs} = E^c \oplus E^{ss}$ the strong unstable, strong stable, center unstable, and center stable bundle, respectively. We also set $E^{su} = E^{ss} \oplus E^{uu}$. Typically these bundles are only continuous, but they are uniquely integrable [1], giving rise to continuous foliations denoted by W^{uu}, W^{ss}, W^{cu}, and W^{cs}, respectively. Recall that a distribution E is called uniquely integrable (or simply integrable) if it is tangent to a foliation and every differentiable curve everywhere tangent to E is wholly contained in a leaf of the foliation. If the flow is $C^{1+\text{Hölder}}$, then these foliations are in fact absolutely continuous (§3-C). By a classical result of Anosov [1], Anosov flows are also structurally stable and, if they preserve a C^1 volume form, ergodic.

If $\dim E^{uu} = 1$, we call the Anosov flow of codimension one [16]. (The assumption $\dim E^{ss} = 1$ works just as well, since we can reverse the direction of the flow.) Verjovsky [32, 33, 34] showed that if $\dim M > 3$, then codimension one Anosov flows are topologically transitive and the universal covering space of M is $\mathbb{R}^n$.

Regularity. In general, the bundles E^{uu}, E^{ss}, E^{cs}, and E^{cu} are only Hölder continuous (see [14]). The Hölder invariant section theorem (see [14, 27, 25]) implies that if two Anosov vector fields are C^1 close, then the Hölder exponents of their strong stable bundles are close. That is, the Hölder exponent $\theta(X)$ of the strong stable bundle E^{ss} for X varies continuously with X in the C^1 topology.

The foliations W^{uu}, W^{ss}, W^{cs}, and W^{cu} are Hölder (cf., [25]) in the sense that their holonomy maps are uniformly Hölder, but their leaves are as smooth as the flow. If the flow is of codimension one and $n > 3$, then E^{cs} is of class $C^{1+\theta}$, for some $\theta \in (0, 1)$. If in addition it preserves a C^1 volume form, then E^{uu} is also of class $C^{1+\theta}$, for some $\theta \in (0, 1)$ (cf., [14]). This implies that their holonomies are $C^{1+\theta}$; see [25].

A general note on regularity of foliations is in order here. Following [25], the usual variants of the definition of a C^r foliation are:

(a) the leaves are tangent to a C^r distribution;
(b) the foliation charts are C^r diffeomorphisms;

(c) the leaves and the local holonomy maps along them are uniformly C^r.

In this paper, we use (a). Since the center stable and strong unstable distributions are $C^{1+\theta}$, the relevant value of r is $1+\theta$. According to [25], when $r = 1+\theta$, $0 \leq \theta \leq \text{Lip}$, the relations among the above definitions are: (a) $\Rightarrow$ (b), (b) $\not\Rightarrow$ (a), and (b) $\Leftrightarrow$ (c). (More can be said: by Hart's smoothing theorem, a foliation satisfying (b) is diffeomorphic by an ambient C^r diffeomorphism, to a foliation satisfying (a).) Therefore, for E^{cs} and E^{uu}, statements (a), (b) and (c) are all true.

Cross sections. Recall that a smooth compact codimension one submanifold Σ of M is called a (global) cross section for a flow if it intersects every orbit transversely. If this is the case, then every point $p \in \Sigma$ returns to Σ, defining the Poincaré or first-return map $g : \Sigma \to \Sigma$. The flow can then be reconstructed by suspending g under the roof function equal to the first-return time [12, 15, 26].

Existence of global cross sections to Anosov flows was studied by Plante in his Ph.D. thesis. He showed:

Plante's Theorem ([20]). *Let $\{f_t\}$ be an Anosov flow.*

(a) *If E^{su} is integrable, then the flow admits a smooth global cross section.*
(b) *If the flow is of codimension one and E^{su} is integrable, then every leaf of the corresponding foliation is a global cross section with* constant *first-return time.*
(c) *E^{su} is integrable if and only if the foliations W^{ss} and W^{uu} are jointly integrable.*

Foliations W^{ss} and W^{uu} are jointly integrable if in every foliation chart for W^{ss} and W^{uu}, the W^{uu}-holonomy (§3-C) takes W^{ss}-plaques to W^{ss}-plaques. The opposite situation is that of *su*-accessibility, where any two points of M can be connected by a continuous path consisting of finitely many smooth arc alternately in W^{ss} and W^{uu} [24].

Plante's Theorem will be the main tool for proving the existence of a global cross section. Note that if the first-return time is constant, then the periods of all periodic points are rationally dependent. This property is clearly not robust, since it can be destroyed by a small non-trivial time-change. Reparametrization will consequently play an important role in the proof.

3. PRELIMINARIES

This section contains preparatory results on Lyapunov exponents, synchronization, holonomy, *su*-disks, regularization, and the key estimate.

3-A. **Lyapunov Exponents.** Let $\Phi = \{f_t\}$ be a C^1 flow on a compact manifold M. For $x \in M$ and $v \in T_xM \setminus \{0\}$, recall that the Lyapunov exponent of v is defined by

$$\chi = \lim_{t\to\infty} \frac{1}{t} \log\|T_x f_t(v)\|. \tag{$\diamond$}$$

This means that $\|T_x f_t(v)\| \sim e^{\chi t}\|v\|$, as $|t| \to \infty$. If this limit exists, the set of vectors in T_xM (including zero) with the same Lyapunov exponent χ is a vector subspace of T_xM, which we call the Lyapunov space of χ and denote by $E^\chi(x)$. The fundamental properties of Lyapunov exponents and their Lyapunov spaces are described by the celebrated

Oseledets's Multiplicative Ergodic Theorem ([5, 18]). *Suppose that $\Phi = \{f_t\}$ is a C^1 flow preserving a Borel probability measure μ on a compact manifold M. Then there exists a set $\mathscr{R} \subset M$ of full measure such that every point in $\mathscr{R}$ is* Lyapunov regular. *This means that for every $x \in \mathscr{R}$ there exists a splitting, called the* Oseledets splitting *of Φ,*

$$T_xM = \bigoplus_{i=1}^{\ell(x)} E_i(x), \tag{3.1}$$

and numbers $\chi_1 < \cdots < \chi_\ell$ such that:

(a) *The bundles* E_i *are* Φ*-invariant,*

$$T_x f_t(E_i(x)) = E_i(f_t x),$$

and depend Borel measurably on x*.*

(b) *For all* $v \in E_i(x) \setminus \{0\}$,

$$\lim_{|t|\to\infty} \frac{1}{t} \log\|T_x f_t(v)\| = \chi_i(x),$$

that is, $E_i(x) = E^{\chi_i}(x)$*. The convergence is uniform on the unit sphere in* $E_i(x)$*.*

(c) *For for any* $I, J \subset \{1, \dots, \ell(x)\}$ *with* $I \cap J = \emptyset$*, the angle function is tempered, i.e.,*

$$\lim_{|t|\to\infty} \frac{1}{t} \log \sphericalangle(T_x f_t(E_I(x)), T_x f_t(E_J(x))) = 0,$$

where $E_I = \bigoplus_{i\in I} E_i$*.*

(d) *For every* $x \in \mathscr{R}$*,*

$$\lim_{|t|\to\infty} \frac{1}{t} \log\det T_x f_t = \sum_{i=1}^{\ell(x)} \chi_i(x) \dim E_i(x).$$

(e) *There is a corresponding decomposition of the cotangent bundle,*

$$T_x^* M = \bigoplus_{i=1}^{\ell(x)} E_i^*(x).$$

The bundles E_i^* *depend Borel measurably on* $x \in \mathscr{R}$ *and are* Φ*-invariant in the sense that*

$$T_x' f_t(E_i^*(x)) = E_i^*(f_t x),$$

where

$$T_x' f_t = (T_x^* f_t)^{-1} : T_x^* M \to T_{f_t x}^* M$$

is the inverse of the codifferential $T_x^* f_t = (T_x f_t)^*$ *of* f_t*.*

(f) *If* Φ *is ergodic with respect to* μ*, then the functions* ℓ *and* χ_i *are* μ*-almost everywhere constant.*

One can also speak of *forward* (or positive) and *backward* (or negative) regularity, where one considers only $t \to +\infty$ or $t \to -\infty$, respectively.

Note that the Oseledets splitting need not be defined on the whole manifold nor do the above limits have to be uniform. However, it turns out that for large set of systems one *can* expect a certain amount of uniformity, in the sense explained below.

Definition (Dominated splitting)**.** *For a diffeomorphism* $f : M \to M$*, we say that a* Tf*-invariant splitting* $T_\Lambda M = E \oplus F$ *over an* f*-invariant set* Λ *is* dominated *(denoted by* $E \prec F$*) if there exists an* $n \in \mathbb{N}$ *and a constant* $\sigma < 1$ *such that for all* $x \in \Lambda$*,*

$$\|T_x f^n\restriction_{E(x)}\| \leq \sigma m(T_x f^n\restriction_{F(x)}).$$

Here $m(L)$ denotes the minimum norm of a linear transformation L: $m(L) = \inf\{\|Lv\| : \|v\| = 1\}$. This means that for $v \in T_\Lambda M \setminus (E \cup F)$, the forward iterates of v converge to F and its backward iterates converge to E. The definition for flows is analogous.

More generally, we say that a splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ into an arbitrary number of invariant subbundles is dominated if for every $1 \leq i < k$,

$$E_1 \oplus \cdots \oplus E_i \prec E_{i+1} \oplus \cdots \oplus E_k.$$

Subsequently, when talking about a dominated splitting, we will always be referring to this general definition of the term.

Bochi and Viana [7] showed that there exists a residual (dense G_δ) set $\mathscr{D}$ in the space of C^1 volume preserving diffeomorphisms of M such that for every $f \in \mathscr{D}$ and almost every point x, either (a) all Lyapunov exponents of f are zero at x, or (b) the Oseledets splitting of f is dominated on the orbit of x. If f is ergodic, this means that either (a) all exponents vanish at almost every point or (b) the Oseledets splitting extends *continuously* to a dominated splitting on the whole manifold [7].

The results of Bessa [6] for volume preserving non-singular flows in dimension three and Bochi-Viana [7] for volume preserving diffeomorphisms extend to volume preserving Anosov flows in any dimension [35]. Namely, for the C^1 generic volume preserving Anosov flow, the Oseledets splitting is dominated and extends continuously over the whole underlying manifold (the other alternative in the dichotomy does not apply, since an Anosov flow cannot have all its Lyapunov exponent equal to zero).

Let $\Phi = \{f_t\}$ now be a volume preserving codimension one Anosov flow on M, $n > 3$, and let (3.1) be the Oseledets splitting relative of the reverse flow f_{-t}. Recall that Φ is topologically transitive [34], hence ergodic relative to Lebesgue measure, so ℓ and χ_i's are a.e. constant functions. Observe that $\chi_1 < 0$, $\chi_2 = \chi(X) = 0$, $\chi_i > 0$, for $3 \leq i \leq \ell$, and $E^{ss}(x) = E_3(x) \oplus \cdots \oplus E_\ell(x)$. Let

$$F_k = \bigoplus_{i=3}^{k} E_i. \tag{3.2}$$

The above discussion implies that for the C^1-generic $\{f_t\}$, the bundles F_k are continuous.

Oseledets Regularity functions. For a fixed Lyapunov exponent $\chi = \chi_i$ corresponding to the Lyapunov bundle $E = E_i$, $\varepsilon > 0$, and $x \in \mathscr{R}$, define $R_\varepsilon(x)$ to be the infimum of all numbers $R \geq 1$ such that the following inequalities hold for all $t \geq 0$:

$$\begin{aligned} R^{-1} e^{(\chi-\varepsilon)t} &\leq \|T_x^E f_t\| \leq R e^{(\chi+\varepsilon)t}, \\ R^{-1} e^{(-\chi-\varepsilon)t} &\leq \|T_x^E f_{-t}\| \leq R e^{(-\chi+\varepsilon)t}, \\ R^{-1} e^{(-\chi-\varepsilon)t} &\leq \|T_x^E f_t'\| \leq R e^{(-\chi+\varepsilon)t}, \\ R^{-1} e^{(\chi-\varepsilon)t} &\leq \|T_x^E f_{-t}'\| \leq R e^{(\chi+\varepsilon)t}. \end{aligned}$$

Here $T^E f_t$ denotes the restriction of Tf_t to E and $T^E f_t' = \left\{(T^E f_t)^*\right\}^{-1}$. We refer to $R_\varepsilon : \mathscr{R} \to [1, \infty)$ as an Oseledets regularity function (relative to χ and ε), or simply a regularity function (we borrowed the name from [23]; see also [5]).

An immediate corollary of the definition of R_ε is that

$$R_\varepsilon(x)^{-1} e^{-\varepsilon t} \leq \frac{\|T_x^E f_t\|}{e^{\chi|t|}} \leq R_\varepsilon(x) e^{\varepsilon t}, \tag{3.3}$$

for all $x \in \mathscr{R}$ and $t \in \mathbb{R}$. The following result was proved in [28].

3.1. **Theorem.** *If E is continuous on the entire manifold M, then for every $\varepsilon > 0$ there exists an open set V_ε of full measure in M such that R_ε is locally bounded on V_ε.*

Sketch of proof. Since E is continuous on M, it follows that R_ε is lower semicontinuous on M, as the supremum of a collection of continuous functions. This implies that the sets $H_k = \{x \in M : R_\varepsilon(x) \leq k\}$ are closed. Since their union equals M, by the Baire category theorem at least one of them has nonempty interior, hence contains an open set U. Then $V_\varepsilon = \bigcup_{t \in \mathbb{R}} f_t(U)$ is open and has full measure, by ergodicity. Furthermore, R_ε is locally bounded on it. This follows from the fact that R_ε is a slowly varying function: $\sqrt{R_\varepsilon(x)} e^{-\varepsilon|t|} \leq R_\varepsilon(f_t x) \leq R_\varepsilon(x)^2 e^{2\varepsilon|t|}$, for all x and t. For details, see [28]. □

If Φ is a volume preserving codimension one Anosov flow, for each $3 \leq k \leq \ell$ and $\varepsilon > 0$, we can also consider the regularity function R_ε^k responsible for the bundle F_k defined in (3.2), relative to the reverse flow f_{-t}. This function is defined by the requirement that it satisfy (3.3) with E, t replaced by $F_k, -t$, respectively. An argument analogous to that in the proof of Theorem 3.1 yields the following

3.2. **Corollary.** *If F_k is continuous on M, then for each $\varepsilon > 0$ there exists an open set of full measure on which R^k_ε is locally bounded.*

3-B. **Synchronization.** In this section we show how to reparametrize an Anosov flow to obtain another Anosov flow with $\det T_x f_t \restriction_{E^{uu}} \equiv e^{ct}$, where $c > 0$ is a constant. This technique is called synchronization and was first described by Parry in [19] who used it to obtain a system for which the SRB measure coincides with the measure of maximal entropy. A similar result, with mildly different assumptions, was proved in [30]. The construction goes as follows.

Let $\{f_t\}$ be a transitive C^r ($r = k+\text{Hölder}$, $k \geq 2$) Anosov flow on M such that E^{cs} and E^{uu} are of class $C^{1+\theta}$, for some $0 < \theta < 1$. This is the case if the flow is of codimension one and preserves a C^1 volume form on a manifold of dimension > 3, which we now assume. Without loss of generality, we may also assume that E^{uu} is orientable. (Otherwise, pass to a double cover of M.) Let Y be a $C^{1+\theta}$ unit vector field generating E^{uu}; its flow is denoted by $\{\phi_t\}$ throughout the paper. Let $\lambda(x,t) = \det T_x f_t \restriction_{E^{uu}}$ and define

$$\psi(x) = \frac{d}{dt}\bigg|_{t=0} \log \lambda(x,t). \tag{3.4}$$

It is not hard to see (cf., [30]) that ψ is of class $C^{1+\text{Hölder}}$ and that there exists a Riemann structure $\mathcal{R}_*$ on M with respect to which $\psi > 0$. This Riemann structure is as smooth as E^{uu} and E^{cs}, i.e., $C^{1+\theta}$. Reparametrize X by

$$\tilde{X} = \frac{1}{\psi} X.$$

It is a well known theorem of Anosov and Sinai [2] that $\tilde{X}$ generates an Anosov flow $\{\tilde{f}_t\}$. Furthermore [30],

$$\det T_x \tilde{f}_t \restriction_{\tilde{E}^{uu}} \equiv e^t,$$

where $\tilde{E}^{uu}$ denotes the strong unstable bundle of the new flow.

Definition. *The reparametrized flow $\{\tilde{f}_t\}$ is called the synchronization of $\{f_t\}$.*

Reparametrization alters the strong bundles but does not change the center bundles, i.e., $\tilde{W}^{cs} = W^{cs}$ and $\tilde{W}^{cu} = W^{cu}$. The new strong unstable bundle $\tilde{E}^{uu}$ can be expressed as (cf., [19])

$$\tilde{E}^{uu} = \{w + \xi(w)X : w \in E^{uu}\},$$

where ξ is a continuous 1-form on E^{uu} defined by

$$\xi_x(w) = \frac{1}{\psi(x)} \int_0^\infty d(\psi \circ f_{-t})(w)\, dt. \tag{3.5}$$

There is an analogous characterization of the strong stable bundle $\tilde{E}^{ss}$: there exists a continuous 1-form η on E^{ss} such that

$$\tilde{E}^{ss} = \{v + \eta(v)X : v \in E^{ss}\}. \tag{3.6}$$

Let us look at the regularity of $\tilde{E}^{uu}$ more closely. The synchronized flow is only $C^{1+\text{Hölder}}$, so we cannot use the C^1-Section Theorem [14] to show that $\tilde{E}^{uu}$ is C^1.

However, we know that W^{uu} is of class $C^{1+\theta}$ and has leaves as smooth as the system, i.e., C^r. Recall that the adapted Riemann structure $\mathcal{R}_*$ is required to have the following properties [30]: (i) E^{uu} is orthogonal to E^{cs} relative to $\mathcal{R}_*$; (ii) $\mathcal{R}_*$ coincides with the original Riemann structure on E^{cs}. Thus we can assume that along the W^{uu}-leaves, $\mathcal{R}_*$ is as smooth as the flow, i.e., C^r. Recall that $\lambda(x,t)$ is the Jacobian determinant of the C^r map f_t between C^r leaves of the C^1 foliation

W^{uu}. This implies that $x \mapsto \lambda(x,t)$ is C^1 and $t \mapsto \lambda(x,t)$ is C^r; however, in the W^{uu}-direction, $x \mapsto \lambda(x,t)$ is as smooth as Tf_t, i.e., C^{r-1}. By (3.4), ψ is C^{r-1} in the W^{uu}-direction. Thus if $r \geq 3$, then $d\psi(Y)$ is at least of class C^1. By (3.5), we have

$$\begin{aligned}\xi(Y(x)) &= \frac{1}{\psi(x)} \int_0^\infty d(\psi \circ f_{-t})(Y(x))\, dt \\ &= \frac{1}{\psi(x)} \int_0^\infty d\psi(Tf_{-t}(Y(x)))\, dt \\ &= \frac{1}{\psi(x)} \int_0^\infty \lambda(x,-t)\, d\psi(Y(f_{-t}x))\, ds,\end{aligned}$$

which implies that $\xi(Y)$ is C^1. Therefore, $\tilde{Y} = Y + \xi(Y)X$ is C^1, so we have the following

3.3. **Lemma.** *The strong unstable bundle $\tilde{E}^{uu}$ of the synchronized flow is of class C^1.*

If $\{f_t\}$ preserves a C^1 volume form Ω, then $\{\tilde{f}_t\}$ preserves $\tilde{\Omega} = \psi\Omega$. However, $\tilde{\Omega}$ does not have to equal the volume form defined by the adapted Riemann structure $\mathcal{R}_*$, but since Lyapunov exponents are independent on the Riemann structure, this makes no difference in the subsequent analysis.

The flow $\{\tilde{f}_t\}$ is $C^{1+\text{Hölder}}$, so the Hölder invariant section theorem applies and guarantees that its strong stable foliation is Hölder [14, 27, 25]. Furthermore, we have:

3.4. **Proposition.** *Suppose $\{\tilde{f}_t\}$ is the synchronization (or, more generally, a C^1 reparametrization) of a volume preserving codimension one Anosov flow $\{f_t\}$ on M, $n > 3$. Let $\{\chi_1, \ldots, \chi_\ell\}$, $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{\tilde{\ell}}\}$ be the Lyapunov exponents of $f_{-t}, \tilde{f}_{-t}$ corresponding to Oseledets decompositions $\bigoplus E_i$, $\bigoplus \tilde{E}_j$ over regular sets $\mathscr{R}, \tilde{\mathscr{R}}$, respectively. Then $\tilde{\mathscr{R}} = \mathscr{R}$ and:*

- (a) *For every $x \in \mathscr{R}$ and all $3 \leq i \leq \ell$,*

$$\tilde{E}_i(x) = \{v + \eta(v)X : v \in E_i(x)\},$$

where X is the infinitesimal generator of $\{f_t\}$ and η is the 1-form in (3.6). In particular, $\tilde{E}_i$ and E_i have the same dimension and $\tilde{\ell} = \ell$.

- (b) *There exists a constant C such that $\tilde{\chi}_i = C\,\chi_i$, for all $1 \leq i \leq \ell$.*

Proof. Since $\{f_t\}$ and $\{\tilde{f}_t\}$ have the same orbits, there exists a C^1-function $\varrho : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}_t(x) = f_{\varrho(x,t)}(x).$$

It is not hard to see that

$$\varrho(x,t) = \int_0^t \frac{ds}{\psi(\tilde{f}_s x)},$$

where $\psi : M \to \mathbb{R}_+$ is the C^1 time-change.

Let x be a regular point for f_t and let $v \in E_i(x) \setminus \{0\}$. Set $\tilde{v} = v + \eta(v)X$. Then

$$\begin{aligned}T\tilde{f}_t(\tilde{v}) &= Tf_{\varrho(x,t)}(\tilde{v}) + \frac{\partial \varrho}{\partial x}(\tilde{v})X \\ &= Tf_{\varrho(x,t)}(v) + \left\{\eta(v)\left[1 + \frac{\partial \varrho}{\partial x}(X)\right] + \frac{\partial \varrho}{\partial x}(v)\right\} X.\end{aligned}$$

Since $T\tilde{f}_t(\tilde{v}) \in \tilde{E}^{ss}$ and $Tf_{\varrho(x,t)}(v) \in E^{ss}$, it follows that

$$T\tilde{f}_t(\tilde{v}) = Tf_{\varrho(x,t)}(v) + \eta(Tf_{\varrho(x,t)}(v))X.$$

Thus $\|T\tilde{f}_t(\tilde{v})\| \sim \|Tf_{\varrho(x,t)}(v)\| \sim e^{\varrho(x,t)\chi_i}\|v\|$, as $t \to \pm\infty$, so

$$\tilde{\chi}(\tilde{v}) = \lim_{t \to -\infty} \frac{\varrho(x,t)}{t}\chi_i = C\chi_i,$$

where $C = \int_M (1/\psi)$, by Birkhoff's Ergodic Theorem. Therefore, $x \in \tilde{\mathscr{R}}$, and $\tilde{v}$ belongs to the Lyapunov space for $\tilde{f}_{-t}$ corresponding to $\tilde{\chi}_i = C\,\chi_i$. □

3.5. **Corollary.** *If the Oseledets splitting of* $\{f_t\}$ *is continuous on all of* M*, then so is that of its synchronization* $\{\tilde{f}_t\}$.

Now let us drop the tildes and assume $\{f_t\}$ is synchronized. Then we have the following characterization of the Lyapunov exponents of f_{-t}.

3.6. **Proposition.** *Suppose* $\{f_t\}$ *is a synchronized volume preserving codimension one Anosov flow and* $n > 3$.

(a) *If* $n = 4$*, then* $\chi_3 \leq 1/2$*. If* $\dim E_3 = 2$*, then* $\chi_\ell = 1/2$.

(b) *If* $n > 4$*, then* $\chi_{\ell-1} + \chi_\ell < 1$*. In particular,* $\chi_{\ell-1} \leq 1/2$*. If in addition,* $\dim E_\ell > 1$*, then* $\chi_\ell < 1/2$.

Proof. Note first that $\chi_1 = \chi(Y) = -1$. Then by part (d) of the Oseledets's Multiplicative Ergodic Theorem,

$$\sum_{i=3}^{\ell} \chi_i \dim E_i = 1,$$

which easily implies (a). If $n > 4$, then $1 \geq \chi_{\ell-1}\dim E_{\ell-1} + \chi_\ell \dim E_\ell \geq \chi_{\ell-1} + \chi_\ell$, where at least one inequality is strict. This yields (b). □

Standing Assumption. Unless stated otherwise, in the remainder of the paper all flows are assumed to be synchronized, volume preserving, codimension one and Anosov on a C^∞ closed Riemannian manifold of dimension $n > 3$.

3-C. **Holonomy and** su**-Disks.** If $\mathscr{F}$ is a continuous foliation with C^1 leaves, we define its holonomy as follows. Fix a foliation chart U, a point $p \in U$, and q in the plaque $\mathscr{F}_U(p)$. Choose C^1 disks D_p, $D_q \subset U$ transverse to $\mathscr{F}$, with $p \in D_p$, $q \in D_q$. Then the holonomy of $\mathscr{F}$ relative to D_p, D_q is the map $\mathbf{h}^{\mathscr{F}} : D_p \to D_q$ defined by sliding points along the plaques of $\mathscr{F}$. Namely, for $x \in D_p$, $\mathbf{h}^{\mathscr{F}}(x) = y$ if $\{y\} = \mathscr{F}_U(x) \cap D_q$. This defines a homeomorphism between D_p and a subset of D_q. If $T\mathscr{F}$ is C^1, then so is $\mathbf{h}^{\mathscr{F}}$ [25]. Denote by $\mathrm{Jac}_x(\mathbf{h}^{\mathscr{F}})$ the Jacobian determinant of $\mathbf{h}^{\mathscr{F}}$ at x. Recall that for a linear isomorphism $T : V \to W$ between inner product spaces, the determinant of T is the volume of the parallelepiped spanned by $T(e_1), \ldots, T(e_k)$, where $\{e_1, \ldots, e_k\}$ is an orthonormal basis for V. If for every choice of U, D_p, and D_q, $\mathbf{h}^{\mathscr{F}}$ sends sets of measure zero in D_p to sets of measure zero in D_q, we say that $\mathscr{F}$ is absolutely continuous. Then by the Radon-Nikodym theorem, $\mathrm{Jac}(\mathbf{h}^{\mathscr{F}})$ is well-defined. It is a classical result of Anosov [1] that the invariant foliations of a $C^{1+\text{Hölder}}$ Anosov system are absolutely continuous. In our case, foliations under consideration, W^{cs} and W^{uu}, are $C^{1+\theta}$, so the holonomy is actually continuously differentiable in the usual sense [25].

Let $U \subset M$ now be a foliation chart for both W^{cs} and W^{uu}. If $p \in U$, $q \in W^{uu}_{\mathrm{loc}}(p)$, and $x \in W^{ss}_{\mathrm{loc}}(p)$, the uu- and cs-holonomy are C^1 maps

$$\mathbf{h}^{uu}_{p,q} : W^{cs}_{\mathrm{loc}}(p) \to W^{cs}_{\mathrm{loc}}(q), \qquad \mathbf{h}^{cs}_{p,x} : W^{uu}_{\mathrm{loc}}(p) \to W^{uu}_{\mathrm{loc}}(x).$$

Similarly, we can define the cu-holonomy $\mathbf{h}^{cu}_{p,q} : W^{ss}_{\mathrm{loc}}(p) \to W^{ss}_{\mathrm{loc}}(q)$. Note that if the unstable manifolds W^{uu} are parametrized by the flow $\{\phi_t\}$ of $Y \in E^{uu}$, then $\mathbf{h}^{cs}_{p,x}$ can be regarded as a map between intervals of real numbers. For simplicity, we will later make a slight abuse of notation and identify these two versions of $\mathbf{h}^{cs}_{p,x}$.

For a simple C^1 path $\gamma : [0,1] \to W^{ss}_{\mathrm{loc}}(p)$ from p to x, define a closed piecewise C^1 path Γ as follows. Let Γ be the sum of $-\gamma$, the uu-arc $[p,q]_{uu}$ from p to q, the C^1-path $\mathbf{h}^{uu}_{p,q}(\gamma)$, and the

uu-arc $[\mathbf{h}^{uu}_{p,q}(x), x]_{uu}$ from $\mathbf{h}^{uu}_{p,q}(x)$ to x (see Fig. 1). Let D_γ be the 2-disk foliated by W^{uu} whose boundary is Γ.

Definition. *$D = D_\gamma$ is called an su-*disk *with* base *γ.*

Further let $(\partial D)^{cs} = \mathbf{h}^{uu}_{p,q}(\gamma) - \gamma$ and $(\partial D)^{uu} = [p,q]_{uu} - [x, \mathbf{h}^{uu}_{p,q}(x)]_{uu}$ be the cs- and uu-component of ∂D.

Define a 1-form ω by requiring $\mathrm{Ker}(\omega) = E^{cs}$ and $\omega(Y) = 1$. It was shown in [30] that (for a synchronized flow)

$$d\omega = \alpha \wedge \omega,$$

where α is the 1-form defined in (1.1).

Recall a result from foliation theory (see, for instance, Exercise 2.3.16 on p.66 in [9] as well as (17.20) in [1]): let $\mathscr{F}$ be a C^1 codimension one foliation such that:

(i) $T\mathscr{F} = \mathrm{Ker}(\omega)$, for some a C^1 1-form ω;
(ii) $d\omega = \alpha \wedge \omega$, for some continuous 1-form α;
(iii) p_0, p_1 lie in a same plaque of $\mathscr{F}$;
(iv) Σ_i is a transversal for $\mathscr{F}$ passing through p_i $(i = 0, 1)$ and $h : \Sigma_0 \to \Sigma_1$ is the corresponding holonomy map of $\mathscr{F}$;
(v) $x_0 \in \Sigma_0$ and $h(x_0) = x_1 \in \Sigma_1$.
(vi) σ is a C^1 path in the leaf $\mathscr{F}(x_0)$ connecting x_0 and x_1.

Then $\log h'(x_0) = \int_\sigma \alpha$. Since $\int_{\mathbf{h}^{uu}_{p,q}(\gamma)} \alpha = \int_{\partial D} \alpha$, this yields

$$\log \mathrm{Jac}_q(\mathbf{h}^{cs}_{p,x}) = \int_{\partial D} \alpha. \tag{3.7}$$

We now have the following characterization of joint integrability.

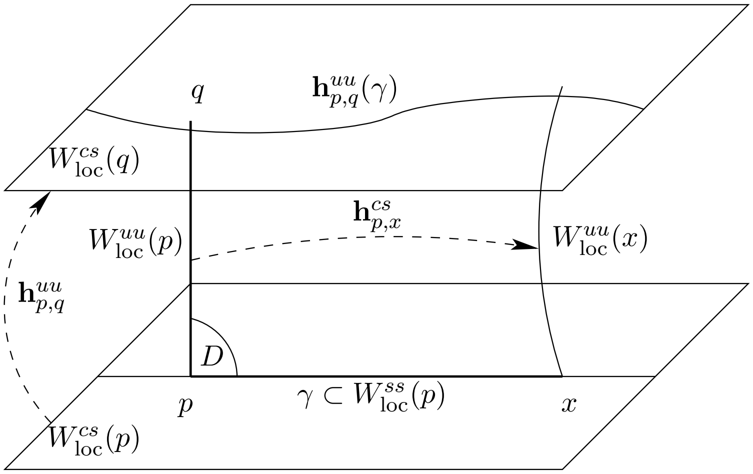

Figure 1. An su-disk D with base γ.

3.7. **Proposition.** *The following statements are equivalent.*[1]

(a) *W^{ss} and W^{uu} are jointly integrable.*
(b) *$T_x \mathbf{h}^{uu}_{p,q}(E^{ss}) = E^{ss}$, for all $p \in M$, $q \in W^{uu}_{\mathrm{loc}}(p)$, and $x \in W^{ss}_{\mathrm{loc}}(p)$.*
(c) *$\int_{\partial D} \alpha = 0$, for every small su-disk D.*
(d) *$\mathrm{Jac}_q(\mathbf{h}^{cs}_{p,x}) = 1$, for all $p \in M$, $x \in W^{ss}_{\mathrm{loc}}(p)$, and $q \in W^{uu}_{\mathrm{loc}}(p)$.*
(e) *$\mathrm{Jac}_x(\mathbf{h}^{cu}_{p,q}) = 1$, for all $p \in M$, $x \in W^{ss}_{\mathrm{loc}}(p)$, and $q \in W^{uu}_{\mathrm{loc}}(p)$.*

---

[1] Parts (d) and (e) are not used in the remainder of the paper.

Proof. Equivalence of (a) and (b) is clear enough. Part (a) implies (c) by definition, since $E^{ss} \subset \operatorname{Ker}(\alpha)$. Assume (c) and let D be a small *su*-disk as above. Recall that its base is the path $\gamma : [0,1] \to W^{ss}_{\text{loc}}(p)$. For $0 < s \leq 1$, set $\gamma_s = \gamma \restriction_{[0,s]}$. Then, by assumption, $\int_{\mathbf{h}^{uu}_{p,q}(\gamma_s)} \alpha = 0$, for all $0 < s \leq 1$. This means that $\mathbf{h}^{uu}_{p,q}(\gamma)$ is entirely contained in $W^{ss}_{\text{loc}}(q)$. Therefore, W^{ss} and W^{uu} are jointly integrable.

Parts (c) and (d) are equivalent by (3.7). To show that (d) and (e) are equivalent, we use the following fact, proved in [30]. If Ω is a C^1 volume form preserved by the flow and $\Theta = i_X i_Y \Omega$, where i_X denotes the contraction by X, then $\operatorname{Ker}(\Theta) = E^{cu}$ and $d\Theta = -\alpha \wedge \Theta$. Therefore, by an analogue of (3.7) for W^{cu} (note that E^{cu} is of class $C^{1+\theta}$), we have

$$\begin{aligned} \log \operatorname{Jac}_x(\mathbf{h}^{cu}_{p,q}) &= \int_{\partial D} (-\alpha) \\ &= -\log \operatorname{Jac}_q(\mathbf{h}^{cs}_{p,x}). \end{aligned}$$

□

Metric properties of *su*-disks. For any $x \in M$, $s \in \mathbb{R}$, and $v \in E^{ss}(x)$, let

$$T\phi_s(v) = a_u(s,v)Y + a_c(s,v)X + Z_s(v),$$

where, as before, $\{\phi_s\}$ is the flow of $Y \in E^{uu}$ and $Z_s(v) \in E^{ss}(\phi_s x)$. (We apologize for the overuse of the letter s.) We will need the following auxiliary result.

3.8. **Lemma.** (a) *For all $x \in M$ and $s \in \mathbb{R}$, $Z_s : E^{ss}(x) \to E^{ss}(\phi_s x)$ is a linear bundle isomorphism covering ϕ_s. The map $s \mapsto Z_s$ is continuous and $Z_0 = \text{identity}$.*

(b) *If $v \in E^{ss}(x)$, then*

$$\lim_{t\to\infty} \frac{\|Tf_{-t}(Z_s(v))\|}{\|Tf_{-t}(v)\|} = 1.$$

In particular, $Z_s(F_{\ell-1}) = F_{\ell-1}$ (cf., (3.2)).

Proof. Part (a) is easy to check. To prove (b), note that since $\{f_t\}$ is synchronized, we have $f_{-t} \circ \phi_s = \phi_{se^{-t}} \circ f_{-t}$, so

$$Tf_{-t}(Z_s(v)) = Z_{se^{-t}}(Tf_{-t}(v)).$$

Therefore,

$$\min_{0 \leq r \leq se^{-t}} \|Z_r\| \leq \frac{\|Tf_{-t}(Z_s(v))\|}{\|Tf_{-t}(v)\|} \leq \max_{0 \leq r \leq se^{-t}} \|Z_r\|.$$

As $t \to \infty$, both the left and right hand side converge to $\|Z_0\| = 1$. □

3.9. **Corollary.** *Let γ be a simple C^1 path in $W^{ss}_{\text{loc}}(p)$ and let $D = D_\gamma$ be an su-disk with base γ. If almost every point of γ is backward Lyapunov regular, then so is almost every point of D. Furthermore, if γ is a.e. tangent to F_k, for some $3 \leq k \leq \ell$, then the Lyapunov exponents of the tangent vectors of $\mathbf{h}^{uu}_{p,q} \circ \gamma$ are $\leq \chi_k$.*

Proof. Let S denote the set of backward Lyapunov regular points in γ. By assumption, S has full measure in γ. Assume $x \in S$ and let $y \in D \cap W^{uu}_{\text{loc}}(x)$ be arbitrary. Then $y = \phi_r(x)$, for some $r \in \mathbb{R}$. Let $v \in T_y M$, $v \neq 0$, be an arbitrary vector. We claim that the limit of $(1/t)\log\|T_y f_{-t}(v)\|$ exists, as $t \to \infty$. It is enough to consider the cases $v \in E^{cu}$ and $v \in E^{ss}$. In the former case, $(1/t)\log\|T_y f_{-t}(v)\|$ converges to 0 or -1, as $t \to \infty$. In the latter one, by Lemma 3.8(b), $\|T_y f_{-t}(v)\|$ is asymptotically equivalent to $\|T_x f_{-t}(Z_{-r}(v))\|$, which converges as $t \to \infty$, since x is a backwards regular point. This proves that y is backward regular. Since almost every point of D is of this form, we obtain the first assertion.

To prove the second one, observe that $(\mathbf{h}^{uu}_{p,q} \circ \gamma)'(r) = T\mathbf{h}^{uu}_{p,q}(\dot\gamma(r)) = a_c X + Z_\rho(\dot\gamma(r))$, for some a_c and ρ depending on r. The Lyapunov exponent of this vector is the greater of the Lyapunov exponents of X and $Z_\rho(\dot\gamma(r))$, which is $\chi(Z_\rho(\dot\gamma(r)))$. By Lemma 3.8(b), $\chi(Z_\rho(\dot\gamma(r)) = \chi(\dot\gamma(r)) \leq \chi_k$ for a.e. r, which completes the proof. □

Given an su-disk $D = D_\gamma$ as above, we parametrize it by

$$\Psi(r,s) = \mathbf{h}^{uu}_{p,\phi_r p}(\gamma(s)),$$

where $0 \leq r \leq \kappa$, for some $\kappa > 0$, and $0 \leq s \leq 1$. Express the W^{uu}-holonomy $\mathbf{h}^{uu}_{p,q}$ as

$$\mathbf{h}^{uu}_{p,q}(x) = \phi_{\mathbf{h}^{cs}_{p,x}(q)}(x), \tag{3.8}$$

where the W^{cs}-holonomy $\mathbf{h}^{cs}_{p,x} : W^{uu}_{\text{loc}}(p) \to W^{uu}_{\text{loc}}(x)$ is regarded as a real-valued function. Since $\mathbf{h}^{uu}_{p,q}$ takes $W^{cs}_{\text{loc}}(p)$ to $W^{cs}_{\text{loc}}(q)$, differentiating with respect to x in the direction of $v \in E^{ss}(x)$, we obtain $d_x \mathbf{h}^{cs}_{p,x}(v) = -a_u(\mathbf{h}^{cs}_{p,x}(q), v)$ and

$$T_x \mathbf{h}^{uu}_{p,q}(v) = a_c(\mathbf{h}^{cs}_{p,x}(q), v) X + Z_{\mathbf{h}^{cs}_{p,x}(q)}(v).$$

Therefore,

$$\frac{\partial \Psi}{\partial s} = a_c(\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p), \dot\gamma(s)) X + Z_{\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p)}(\dot\gamma(s)).$$

It follows from (3.8) that

$$\frac{\partial \Psi}{\partial r} = \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) Y.$$

The area element of D is $\left\| \frac{\partial \Psi}{\partial r} \wedge \frac{\partial \Psi}{\partial s} \right\|$. Recall that $|\partial D|$ denotes the circumference of the boundary of D and $|D|$ its area. Since $f_{-t} \circ \Psi$ is a parametrization of $f_{-t} D$, the area of $f_{-t} D$ can be estimated as follows:

$$\begin{aligned}
|f_{-t} D| &= \iint\limits_{[0,\kappa]\times[0,1]} \left\| T f_{-t} \left(\frac{\partial \Psi}{\partial r} \wedge \frac{\partial \Psi}{\partial s} \right) \right\| \, dr\, ds \\
&= \iint\limits_{[0,\kappa]\times[0,1]} \Big\| T f_{-t} \Big(a_c(\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p), \dot\gamma(s)) \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) X \wedge Y \\
&\qquad + \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) Z_{\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p)}(\dot\gamma(s)) \wedge Y \Big) \Big\| \, dr\, ds \\
&\leq \iint\limits_{[0,\kappa]\times[0,1]} \left| a_c(\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p), \dot\gamma(s)) \right| \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) \| T f_{-t}(X \wedge Y) \| \, dr\, ds \\
&\qquad + \iint\limits_{[0,\kappa]\times[0,1]} \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) \| T f_{-t}(Y \wedge Z(r,s)) \| \, dr\, ds \\
&\leq K \kappa e^{-t} + K \iint\limits_{[0,\kappa]\times[0,1]} \| T f_{-t}(Y \wedge Z(r,s)) \| \, dr\, ds,
\end{aligned} \tag{$\flat$}$$

where $Z(r,s) = Z_{\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p)}(\dot\gamma(s))$ and

$$K = \sup \left\{ \max \left(\left| a_c(\mathbf{h}^{cs}_{p,\gamma(s)}(\phi_r p), \dot\gamma(s)) \right| \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}), \text{Jac}_{\phi_r p}(\mathbf{h}^{cs}_{p,\gamma(s)}) \right) : (r,s) \in [0,\kappa] \times [0,1] \right\}.$$

3.10. **Proposition.** *Let $D = D_\gamma$ be an su-disk as above, with $\gamma \subset W^{ss}_{\text{loc}}(p)$. Then:*

(a) $e^{-t} |\partial f_{-t} D| \to 0$ *and* $|f_{-t} D| \to 0$, *as* $t \to \infty$.

(b) *If γ is tangent to the bundle $F_k \subset E^{ss}$, for some $3 \leq k \leq \ell$, then for every $\varepsilon > 0$,*

$$|\partial f_{-t} D| \leq |\partial D| \, \|R^k_\varepsilon\|_{L^\infty(\partial D)} e^{(\chi_k + \varepsilon)t},$$

and

$$|f_{-t} D| \leq A \|R^k_\varepsilon\|_{L^\infty(D)} e^{(\chi_k + \varepsilon - 1)t},$$

for all $t \geq 0$, where A is a constant depending only on D and the flow.

Remark. The norms $\|R^k_\varepsilon\|_{L^\infty(\partial D)}, \|R^k_\varepsilon\|_{L^\infty(D)}$ may, of course, be infinite.

Proof. (a) If $c : [0,1] \to \partial D$ is a piecewise C^1 parametrization of ∂D, then as $t \to \infty$,

$$\begin{aligned} e^{-t} \left|\partial f_{-t} D\right| &= e^{-t} \int_0^1 \|Tf_{-t}(\dot{c}(s))\| \, ds \\ &= \int_0^1 \|Tf_{-t}(\dot{c}(s) \wedge Y)\| \, ds \\ &\leq \int_0^1 C\nu^{(n-3)t} \|\dot{c}(s) \wedge Y\| \, ds \\ &\leq C\nu^{(n-3)t} \left|\partial D\right|. \end{aligned}$$

The inequality $\|Tf_{-t}(v \wedge Y)\| \leq C\nu^{(n-3)t}\|v \wedge Y\|$, for $v \in E^{ss}$, was proved in [13] (Lemma 1.2) and [30] (Lemma 3.1).

Using $\|Tf_{-t}(Y \wedge Z(r,s))\| \leq C\nu^{(n-3)t}\|Y \wedge Z(r,s)\|$ and ($\flat$), we obtain

$$|f_{-t}D| \leq K\kappa e^{-t} + KC\nu^{(n-3)t} \iint\limits_{[0,\kappa]\times[0,1]} \|Y \wedge Z(r,s)\| \, dr \, ds,$$

which converges to zero, as $t \to \infty$.

(b) Since $\dot{\gamma}$ is tangent to F_k, by Corollary 3.9 the Lyapunov exponents of the tangent vectors to $\mathbf{h}^{uu}_{p,q} \circ \gamma$ are $\leq \chi_k$. Therefore, $\|Tf_{-t}(\dot{c}(s))\| \leq \|R^k_\varepsilon\|_{L^\infty(\partial D)} e^{(\chi_k+\varepsilon)t}\|\dot{c}(s)\|$, for all $t \geq 0$. This yields

$$|\partial f_{-t}D| \leq \int_0^1 \|R^k_\varepsilon\|_{L^\infty(\partial D)} e^{(\chi_k+\varepsilon)t}\|\dot{c}(s)\| \, ds = |\partial D| \, \|R^k_\varepsilon\|_{L^\infty(\partial D)} e^{(\chi_k+\varepsilon)t}.$$

Also by Corollary 3.9.

$$\|Tf_{-t}(Z(r,s))\| \leq R^k_\varepsilon(\Psi(r,s)) e^{(\chi_k+\varepsilon)t}\|Z(r,s)\|.$$

Since $\|Tf_{-t}(Y)\| = e^{-t}$, it follows that

$$\|Tf_{-t}(Y \wedge Z(r,s))\| \leq R^k_\varepsilon(\Psi(r,s)) e^{(\chi_k+\varepsilon-1)t}\|Z(r,s)\|.$$

Using ($\flat$), we obtain

$$|f_{-t}D| \leq K\kappa e^{-t} + K\|R^k_\varepsilon\|_{L^\infty(D)} e^{(\chi_k+\varepsilon-1)t} \iint_{[0,\kappa]\times[0,1]} \|Z(r,s)\| \, drds.$$

Taking $A = 2K\kappa \max\{\|Z(r,s)\| : (r,s) \in [0,\kappa] \times [0,1]\}$, we obtain the second statement in (b). $\square$

3.11. **Corollary.** *Suppose γ is tangent to F_k and $\chi_k < \tau < 1$. If $\|R^k_\varepsilon\|_{L^\infty(D)}$ is finite for some $\varepsilon < \tau - \chi_k$, then*

$$\lim_{t\to\infty} |\partial f_{-t}D|^{1-\tau} \, |f_{-t}D|^\tau = 0.$$

Proof. Let $0 < \varepsilon < \tau - \chi_k$. Then by Proposition 3.10(b),

$$\begin{aligned} |\partial f_{-t}D|^{1-\tau} \, |f_{-t}D|^\tau &\leq \left\{|\partial D| \, \|R^k_\varepsilon\|_{L^\infty(\partial D)} e^{(\chi_k+\varepsilon)t}\right\}^{1-\tau} \left\{A\|R^k_\varepsilon\|_{L^\infty(D)} e^{(\chi_k+\varepsilon-1)t}\right\}^\tau \\ &\leq A^\tau \, |\partial D|^{1-\tau} \, \|R^k_\varepsilon\|_{L^\infty(D)} e^{(\chi_k+\varepsilon-\tau)t} \\ &\to 0, \end{aligned}$$

as $t \to \infty$. $\square$

3-D. **Regularization.** We now review a well known method of approximating locally integrable functions by smooth ones. Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is locally integrable and define its regularization (or mollification) by the convolution $u^\varepsilon = \eta_\varepsilon * u$, where $\eta_\varepsilon(x) = \varepsilon^{-n}\eta\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$, and $\eta : \mathbb{R}^n \to \mathbb{R}$ is the standard mollifier [10, 31]

$$\eta(x) = \begin{cases} A\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

with A chosen so that $\int \eta\, dx = 1$. Note that the support of η_ε is contained in the ball of radius ε centered at 0 and $\int \eta_\varepsilon\, dx = 1$.

3.12. **Proposition.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then:*

(a) $u^\varepsilon \in C^\infty(\mathbb{R}^n)$.
(b) *If $u \in L^\infty$, then $\|u^\varepsilon\|_{L^\infty} \leq \|u\|_{L^\infty}$.*
(c) *If $u \in C^\theta$ $(0 < \theta < 1)$, then $\|u^\varepsilon - u\|_{C^0} \leq \|u\|_{C^\theta}\, \varepsilon^\theta$.*
(d) *If $u \in C^1$, then $\|u^\varepsilon - u\|_{C^0} \leq \|u\|_{C^1}\, \varepsilon$. If u is C^1 along the leaves of a C^1 foliation, then this estimate holds along each leaf.*
(e) *If $u \in C^\theta$, then $\|du^\varepsilon\|_{C^0} \leq \|d\eta\|_{L^1}\|u\|_{C^\theta}\varepsilon^{\theta-1}$, where $\|d\eta\|_{L^1} = \max_i \int_{\mathbb{R}^n} |\partial\eta/\partial x_i|\, dx$.*

Proof. Proof of (a) and (b) can be found in [10] . Although (c)–(e) are probably well known facts, I have not been able to find them in the literature. We therefore sketch their proofs. For (c), we have

$$\begin{aligned} |u^\varepsilon(x) - u(x)| &= \left| \int_{B(0,\varepsilon)} \eta_\varepsilon(y)[u(x-y) - u(x)]\, dy \right| \\ &\leq \|u\|_{C^\theta}\varepsilon^\theta \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, dy \\ &= \|u\|_{C^\theta}\varepsilon^\theta. \end{aligned}$$

If $u \in C^1$, then the same estimates hold with θ replaced by 1. The leafwise version follows straightforwardly. This settles (d).

Observe that since η_ε has compact support,

$$\int_{\mathbb{R}^n} \frac{\partial \eta_\varepsilon}{\partial x_i}(y)\, dy = 0, \tag{3.9}$$

for $1 \leq i \leq n$. Note also that

$$\frac{\partial \eta_\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon^{n+1}} \frac{\partial \eta}{\partial x_i}\left(\frac{x}{\varepsilon}\right).$$

Using this and assuming $u \in C^\theta$, we obtain (e):

$$\begin{aligned}
\left|\frac{\partial u^\varepsilon}{\partial x_i}(x)\right| &= \left|\int_{\mathbb{R}^n} u(x-y)\frac{\partial \eta_\varepsilon}{\partial x_i}(y)\,dy\right| \\
&\overset{\text{by (3.9)}}{=} \left|\int_{B(0,\varepsilon)} [u(x-y)-u(x)]\frac{\partial \eta_\varepsilon}{\partial x_i}(y)\,dy\right|, \\
&\leq \|u\|_{C^\theta}\,\varepsilon^\theta \int_{B(0,\varepsilon)} \left|\frac{\partial \eta_\varepsilon}{\partial x_i}(y)\right|\,dy \\
&= \|u\|_{C^\theta}\,\varepsilon^\theta \int_{B(0,\varepsilon)} \frac{1}{\varepsilon^{n+1}}\left|\frac{\partial \eta}{\partial x_i}\left(\frac{y}{\varepsilon}\right)\right|\,dy \\
&\overset{z=\frac{y}{\varepsilon}}{=} \|u\|_{C^\theta}\,\varepsilon^\theta\cdot\frac{1}{\varepsilon}\int_{B(0,1)} \left|\frac{\partial \eta}{\partial x_i}(z)\right|\,dz, \\
&\leq \|d\eta\|_{L^1}\,\|u\|_{C^\theta}\,\varepsilon^{\theta-1}.
\end{aligned}$$

□

Remark. Regularization on smooth manifolds can be done locally. If $\varphi : U \to \varphi(U) \subset \mathbb{R}^k$ are C^∞ local coordinates, let $\hat{U}$ be an open set whose closure is contained in U. Define $\hat{\varphi} = \varphi\restriction_{\hat{U}}$. Then if $u : U \to \mathbb{R}$ is locally integrable and $\varepsilon > 0$ is small enough, simply take $u^\varepsilon = \left(u\circ\hat{\varphi}^{-1}\right)^\varepsilon \circ \hat{\varphi} : \hat{U} \to \mathbb{R}$.

3-E. **The Key Estimate.** We now derive an upper bound for the integral of α over the boundary of an *su*-disk D in terms of the circumference $|\partial D|$ and area $|D|$. Before we begin, we recall that α is of class C^θ, where $\theta = \theta(X)$ is the Hölder exponent of E^{ss}. However, restricted to any W^{cs}-plaque, α is of class C^1, since $\operatorname{Ker}(\alpha\restriction_{W^{cs}}) = E^{ss}$ is C^1 along the leaves of W^{cs}.

Now fix a finite atlas $\{(U,\varphi)\}$ of M. For each coordinate chart U choose an open set $\hat{U}$ such that the closure of $\hat{U}$ is contained in U and $\{\hat{U}\}$ covers M. Let $\varepsilon_0 = \frac{1}{2}\min_U \inf\{d(x,y) : x \in \partial U, y \in \partial\hat{U}\}$. Then for every chart U and every locally integrable function $u : U \to \mathbb{R}$, the regularization $u^\varepsilon(x)$ is defined for all $\varepsilon \in (0,\varepsilon_0)$ and $x \in \hat{U}$.

Set

$$\|\alpha\|_* = \max_U \left\{\|\alpha\|_{C^\theta(U)}, \sup_P \|\alpha\|_{C^1(P)}\right\},$$

where U is a chart in $\mathscr{A}$ and P runs over all W^{cs}-plaques in U.

Let $\delta > 0$ be the Lebesgue number of the covering $\{\hat{U}\}$. This means that for every set $S \subset M$ with $\operatorname{diam}(S) < \delta$, there exists a coordinate chart U for M such that $S \subset \hat{U}$.

3.13. **Theorem** (The Key Estimate). *Let D be an su-disk. If* $\operatorname{diam}(D) < \delta$ *and* $\frac{|D|}{|\partial D|} < \frac{\varepsilon_0^{2-\theta}}{1-\theta}$, *then*

$$\left|\int_{\partial D}\alpha\right| \leq K(\alpha,\theta)\,|\partial D|^{1-\tau}\,|D|^\tau, \tag{3.10}$$

where $K(\alpha,\theta) = 2\|d\eta\|_{L^1}\|\alpha\|_*[(1-\theta)^\tau + (1-\theta)^{\tau-1}]$ *and* $\tau = \frac{1}{2-\theta}$.

Proof. Since $\operatorname{diam}(D) < \delta$, D is contained in $\hat{U}$, for some coordinate chart U. In U, α can be written as $\sum a_i dx_i$, for some functions $a_i : U \to \mathbb{R}$. These functions inherit properties from α: they are C^θ and on W^{cs}-plaques, they are C^1.

Let $a_i^\varepsilon : \hat{U} \to \mathbb{R}$ be the regularization of a_i defined for $0 < \varepsilon \leq \varepsilon_0$. Set $\alpha^\varepsilon = \sum a_i^\varepsilon dx_i$. Then Proposition 3.12 states that for all $0 < \varepsilon \leq \varepsilon_0$,

$$\|\alpha^\varepsilon - \alpha\|_{C^0} \leq \|\alpha\|_{C^\theta}\,\varepsilon^\theta \qquad \text{and} \qquad \|d\alpha^\varepsilon\|_{C^0} \leq \|d\eta\|_{L^1}\|\alpha\|_{C^\theta}\,\varepsilon^{\theta-1}.$$

Furthermore, since α is C^1 along the plaques of W^{cs} in U, we also have

$$\|(\alpha^\varepsilon - \alpha)\restriction_{W^{cs}}\|_{C^0} \leq \|\alpha\|_*\,\varepsilon. \tag{3.11}$$

Recall that $(\partial D)^{cs}$ is contained in the union of two W^{cs}-plaques, which means that the C^0 distance between α and α^ε along $(\partial D)^{cs}$ is of order ε, as in (3.11). Therefore,

$$\begin{aligned}
\left|\int_{\partial D} \alpha\right| &= \left|\int_{(\partial D)^{cs}} \alpha\right| \\
&\leq \left|\int_{(\partial D)^{cs}} (\alpha - \alpha^\varepsilon)\right| + \left|\int_{(\partial D)^{cs}} \alpha^\epsilon\right| \\
&\leq \|(\alpha - \alpha^\varepsilon)\restriction_{W^{cs}}\|_{C^0}\ |(\partial D)^{cs}| + \left|\int_{\partial D} \alpha^\varepsilon\right| + \left|\int_{(\partial D)^{uu}} (\alpha^\varepsilon - \alpha)\right| \\
&\leq |\partial D|\ \|\alpha\|_* \varepsilon + \left|\int_D d\alpha^\varepsilon\right| + \varepsilon \|\alpha\|_*\, |\partial D| \\
&\leq 2\, |\partial D|\ \|\alpha\|_*\, \varepsilon + |D|\ \|d\alpha^\varepsilon\|_{C^0} \\
&\leq 2\, |\partial D|\ \|\alpha\|_*\, \varepsilon + |D|\ \|d\eta\|_{L^1} \|\alpha\|_*\, \varepsilon^{\theta-1} \\
&\leq 2 \|d\eta\|_{L^1} \|\alpha\|_* \left\{ |\partial D|\ \varepsilon + |D|\ \varepsilon^{\theta - 1} \right\}. \qquad (*)
\end{aligned}$$

Note that the inequality holds for *all* $\varepsilon \in (0, \varepsilon_0)$. Let us minimize the right hand side with respect to ε. It is elementary to check that the function $\varepsilon \mapsto |\partial D|\, \varepsilon + |D|\, \varepsilon^{\theta-1}$ has an absolute minimum equal to

$$B(\theta)\, |\partial D|^{1-\tau}\, |D|^\tau \qquad \text{achieved at} \qquad \varepsilon_* = \left\{ \frac{(1-\theta)\, |D|}{|\partial D|} \right\}^\tau,$$

where $B(\theta) = (1-\theta)^\tau + (1-\theta)^{\tau - 1}$ and $\tau = 1/(2-\theta)$. Observe that ε_* does lie in $(0, \varepsilon_0)$, the permissible range of ε. Therefore, we can take $\varepsilon = \varepsilon_*$ in $(*)$, which yields (3.10). □

Remark. Note that $\tau > 1/2$, for all $\theta \in (0,1)$. Furthermore, since $\theta = \theta(X)$ depends continuously on X in the C^1 topology (see Section 2), so does $\tau = \tau(X)$.

4. Proof of the Main Theorem

We start with an arbitrary C^1 volume preserving codimension one Anosov vector field X_0 on a C^∞ closed Riemannian manifold M of dimension $n > 3$. Recall that, as before, each Oseledets splitting is relative to the reverse flow.

Step 1: Perturbation. Let $\mathscr{U}$ be a C^1-structural stability neighborhood of X_0 such that every vector field in $\mathscr{U}$ is topologically equivalent to X_0. By the work of Bessa [6] and Bochi-Viana [7] (see §3-A), there exists a volume preserving $X_1 \in \mathscr{U}$ such that its flow admits a dominated Oseledets splitting $E_1 \oplus \cdots \oplus E_\ell$ continuous over the whole manifold M. The density result of Arbieto and Matheus [3] gives a C^∞ volume preserving X_2 in $\mathscr{U}$ arbitrarily close to X_1. Since the property of possessing a dominated splitting is open in the C^1 topology, we can assume that X_2 has it. Denote it by $H_1 \oplus \cdots \oplus H_\ell$. The difficulty is that this splitting need not be the *Oseledets* splitting for X_2, since a perturbation can cause some of the Lyapunov bundles for X_1 to split into lower dimensional ones. Observe, however, that since X_1 and X_2 are both codimension one Anosov, E_1, H_1 are the strong unstable bundles and E_2, H_2 are the center bundles.

There are two possibilities:

Case 1: $\dim E_\ell = 1$. Then a perturbation cannot split E_ℓ any further, so H_ℓ is also 1-dimensional and is the top Lyapunov bundle for X_2. It follows that $H_3 \oplus \cdots \oplus H_{\ell-1}$ is the $F_{\ell-1}$-bundle for X_2 and is thus continuous.

Case 2: $\dim E_\ell > 1$. Denote the top Lyapunov exponent of X_i $(i = 1, 2)$ and its synchronization $\tilde{X}_i$ by $\chi_{\text{top}}(X_i)$ and $\chi_{\text{top}}(\tilde{X}_i)$, respectively. If $n > 4$, then $\chi_{\text{top}}(\tilde{X}_1) = \chi_\ell(\tilde{X}_1) < 1/2$

(Proposition 3.6). Thus if X_2 is sufficiently C^1-close to X_1, then $\chi_{\text{top}}(\tilde{X}_2) = \chi_\ell(\tilde{X}_2) < 1/2$ as well. In particular, it is $< \tau(\tilde{X}_2)$.

If $n = 4$, things are a little more subtle. Since the Lyapunov bundle $E_\ell = E_3$ corresponding to $\chi_{\text{top}}(X_1)$ equals two, so does that of the synchronization $\tilde{X}_1$ of X_1 (Proposition 3.4). If X_2 is C^1-close to X_1, then $\tilde{X}_2$ is C^1-close to $\tilde{X}_1$, so $\chi_{\text{top}}(\tilde{X}_2)$ is close to $\chi_{\text{top}}(\tilde{X}_1)$ and $\tau(\tilde{X}_2)$ is close to $\tau(\tilde{X}_1)$. By Proposition 3.6, $\chi_{\text{top}}(\tilde{X}_1) = 1/2 < \tau(\tilde{X}_1)$. This implies that if X_2 is sufficiently C^1-close to X_1, then $\chi_{\text{top}}(\tilde{X}_2) < \tau(\tilde{X}_2)$.

We conclude that it is always possible to find a C^∞ volume preserving $X_2 \in \mathscr{U}$ having one of the following two properties:

(A) The top Lyapunov bundle is has dimension one, and both it and the corresponding $F_{\ell-1}$-bundle are continuous on M.

(B) The top Lyapunov exponent of its synchronization is strictly less than the corresponding number τ.

If X_2 satisfies (A), the remainder of the proof consists of **Steps 2, 3A, 4**, and **5**.

If X_2 satisfies (B), the remainder of the proof consists of **Steps 2, 3B**, and **5**.

Step 2: Synchronization. Let us now synchronize X_2. We obtain a $C^{1+\text{Hölder}}$ Anosov vector field, which we denote by X, with flow $\{f_t\}$. By Propositions 3.4 and 3.6 in §3-B, $\{f_t\}$ has the following properties:

(a) It is volume preserving and of codimension one;

(b) Its center stable bundle E^{cs} and strong unstable bundle E^{uu} are of class $C^{1+\text{Hölder}}$.

(c) The Oseledets splitting for f_{-t}, which we (slightly abusing the notation) denote by $E_1 \oplus \cdots \oplus E_\ell$, and the corresponding Lyapunov exponents satisfy either (A) or (B), where (B) now reads $\chi_\ell < \tau$.

Step 3A. We have an Anosov vector field satisfying (A) and (a)–(c) from Step 2. Since $F_{\ell-1}$ is continuous, by Corollary 3.2, for each $\varepsilon > 0$ there exists an open set G_ε of full measure in M such that the regularity function $R_\varepsilon^{\ell-1}$ is locally bounded on G_ε.

Since $\chi_{\ell-1} \leq \frac{1}{2} < \tau = \tau(X)$, we can pick $\varepsilon > 0$ such that $\varepsilon < \tau - \chi_{\ell-1}$. Let $p \in G_\varepsilon$ and $q \in W^{uu}_{\text{loc}}(p) \cap G$ be arbitrary but fixed. We will show that $T\mathbf{h}^{uu}_{p,q}(F_{\ell-1}) \subset E^{ss}$.

Let $\gamma : [0,1] \to W^{ss}_{\text{loc}}(p)$ be a simple C^1 path tangent to $F_{\ell-1}$ and contained in the set G_ε. Such a path exists, since $F_{\ell-1}$ is continuous. Let $D = D_\gamma$ be the associated su-disk; we assume that q is sufficiently close to p so that $D \subset G_\varepsilon$.

We now use the flow invariance ($f_t^*\alpha = \alpha$) and Theorem 3.13 to estimate the integral of α over $\partial f_{-t}D$, where $t > 0$ is large but fixed for now. To do that, decompose D into k small su-disks D_i of approximately equal size such that Proposition 3.13 can be applied to each $f_{-t}D_i$. We will also make sure that $|f_{-t}D_i| \lesssim \frac{1}{k}|f_{-t}D|$ and $|\partial f_{-t}D_i| \lesssim \frac{1}{k}|\partial f_{-t}D|$, for all $1 \leq i \leq k$.

First, choose an integer k so that, in the notation from §3-C,

$$\frac{2}{\delta}\left|(\partial f_{-t}D)^{cs}\right| < k < \frac{3}{\delta}\left|(\partial f_{-t}D)^{cs}\right|, \tag{4.1}$$

where δ is the Lebesgue number of the covering $\{\hat{U}\}$ defined in §3-E. Divide γ into k segments γ_i (see FIG. 2) so that arcs $f_{-t}(\gamma_i)$ all have equal length. Let $D_i = D_{\gamma_i}$ be the su-disk defined by γ_i.

Observe that for all i, $|(\partial D_i)^{uu}| \leq c\,|(\partial D)^{uu}|$, where $c > 0$ is a constant depending on the diameter of D and the size of $\text{Jac}(\mathbf{h}^{cs}_{p,z})$, as z traverses γ. Without loss we can assume that $c \leq 2$.

Furthermore, $|(\partial f_{-t}D_i)^{cs}| \approx |(\partial f_{-t}D_j)^{cs}|$, for all i, j, so for large enough t,

$$\left|(\partial f_{-t}D_i)^{cs}\right| \leq \frac{2}{k}\left|(\partial f_{-t}D)^{cs}\right| < \delta. \tag{4.2}$$

Since the diameter of $f_{-t}D_i$ is approximately $|(\partial f_{-t}D_i)^{cs}|$, it follows that $\operatorname{diam}(f_{-t}D_i) < \delta$.

Further note that $|\partial f_{-t}D_i| = |(\partial f_{-t}D_i)^{uu}| + |(\partial f_{-t}D_i)^{cs}| < e^{-t}\,|(\partial D_i)^{uu}| + \delta \le 2e^{-t}\,|(\partial D)^{uu}| + \delta$ and $|\partial f_{-t}D| = |(\partial f_{-t}D)^{uu}| + |(\partial f_{-t}D)^{cs}| > e^{-t}\,|(\partial D)^{uu}| + \frac{1}{3}k\delta$. It is not hard to see that this implies

$$|\partial f_{-t}D_i| \le \frac{4}{k}\,|\partial f_{-t}D|\,. \tag{4.3}$$

Moreover, since $|f_{-t}D_i| \approx |f_{-t}\gamma_i| \cdot e^{-t}\,|(\partial D_i)^{uu}| \approx |f_{-t}\gamma_j| \cdot e^{-t}\,|(\partial D_j)^{uu}| \approx |f_{-t}D_j|$, for all i, j, all disks $f_{-t}D_i$ have roughly the same area, so for sufficiently large t and all $1 \le i \le k$,

$$|f_{-t}D_i| \le \frac{2}{k}\,|f_{-t}D|\,. \tag{4.4}$$

To apply Theorem 3.13, it remains to verify that $|f_{-t}D_i|\,/\,|\partial f_{-t}D_i|$ is small. This is indeed the case for large t, as $|f_{-t}D_i|\,/\,|\partial f_{-t}D_i| \approx e^{-t}\delta/[2(\delta + e^{-t})] \approx e^{-t}$.

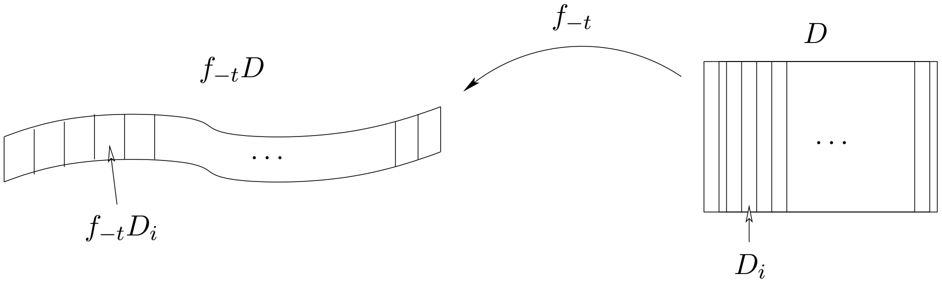

FIGURE 2. Decomposition of D into D_i's.

Therefore,

$$\begin{aligned}
\left|\int_{\partial D}\alpha\right| &= \left|\int_{\partial f_{-t}D}\alpha\right| \\
&\le \sum_{i=1}^{k}\left|\int_{\partial f_{-t}D_i}\alpha\right| \\
&\le \sum_{i=1}^{k} K(\alpha,\theta)\,|\partial f_{-t}D_i|^{1-\tau}\,|f_{-t}D_i|^{\tau} \\
&\le K(\alpha,\theta)\sum_{i=1}^{k}\left(\frac{4}{k}\,|\partial f_{-t}D|\right)^{1-\tau}\left(\frac{2}{k}\,|f_{-t}D|\right)^{\tau} && (\diamond) \\
&\le 4K(\alpha,\theta)\sum_{i=1}^{k}\frac{1}{k}\,|\partial f_{-t}D|^{1-\tau}\,|f_{-t}D|^{\tau} \\
&\le 4K(\alpha,\theta)\,|\partial f_{-t}D|^{1-\tau}\,|f_{-t}D|^{\tau}\,.
\end{aligned}$$

Note that $(\diamond)$ follows from (4.3), $|(\partial D_i)^{uu}| \le 2\,|(\partial D)^{uu}|$, and (4.4). Using (4.1), we arrive to the crucial estimate:

$$\left|\int_{\partial D}\alpha\right| \le 4K(\alpha,\theta)\,|\partial f_{-t}D|^{1-\tau}\,|f_{-t}D|^{\tau}\,. \tag{4.5}$$

Since $D \subset G_\varepsilon$, the norm $\|R_\varepsilon^{\ell-1}\|_{L^\infty(D)}$ is finite, so by Corollary 3.11, $|\partial f_{-t}D|^{1-\tau}\,|f_{-t}D|^{\tau} \to 0$, as $t \to \infty$.

Therefore, $\int_{\partial D}\alpha = 0$ for all *su*-disks $D = D_\gamma$ in G_ε with γ tangent to $F_{\ell-1}$. This implies $T\mathbf{h}_{p,q}^{uu}(F_{\ell-1}) \subset E^{ss}$, for all $p \in G_\varepsilon$ and $q \in W_{\mathrm{loc}}^{uu}(p) \cap G_\varepsilon$. In fact, Lemma 3.8(c) gives $T\mathbf{h}_{p,q}^{uu}(F_{\ell-1}) = F_{\ell-1}$. By continuity, this holds for *all* $p \in M$ and $q \in W_{\mathrm{loc}}^{uu}(p)$. The remainder of the proof consists of Steps 4 and 5.

Step 3B. We have an Anosov vector field satisfying (B) and (a)–(c) from Step 2. In particular $\chi_\ell = \chi_{\text{top}}(X) < \tau = \tau(X)$. Let $0 < \varepsilon < \tau - \chi_\ell$. Since $E^{ss} = F_\ell$ is continuous, by Corollary 3.2 there exists an open set G_ε of full measure on which the regularity function R^ℓ_ε is locally bounded on G_ε. Let D be any *su*-disk contained in G_ε with base γ tangent to E^{ss}. Then estimates completely analogous to those in Step 3A show

$$\left| \int_{\partial D} \alpha \right| \leq 4K(\alpha, \theta) \left| \partial f_{-t} D \right|^{1-\tau} \left| f_{-t} D \right|^{\tau}.$$

Since $D \subset G_\varepsilon$, the norm $\|R^\ell_\varepsilon\|_{L^\infty(D)}$ is finite, so by Corollary 3.11, $\left| \partial f_{-t} D \right|^{1-\tau} \left| f_{-t} D \right|^{\tau} \to 0$, as $t \to \infty$. By the same concluding argument as in Step 3A, it follows that $T\mathbf{h}^{uu}_{p,q}(E^{ss}) = E^{ss}$, for all $p \in M$ and $q \in W^{uu}_{\text{loc}}(p)$. We skip Step 4 and proceed to Step 5.

Step 4. It was shown in Step 3A that $T\mathbf{h}^{uu}_{p,q}(F_{\ell-1}) = F_{\ell-1}$, for all $p \in M$ and $q \in W^{uu}_{\text{loc}}(p)$. In this step we will show that $T_x\mathbf{h}^{uu}_{p,q}$, in fact, takes the *whole* bundle E^{ss} onto itself, for every $p \in M$, $q \in W^{uu}_{\text{loc}}(p)$, and $x \in W^{cs}_{\text{loc}}(p)$. Recall that $\dim E_\ell = 1$, so $F_{\ell-1}$ is of codimension one in E^{ss} (as well as continuous). Set

$$\alpha_s = \phi_s^* \alpha \qquad \text{and} \qquad \alpha_s^\sharp = \alpha_s \restriction_{E^{ss}},$$

where, as before, $\{\phi_s\}$ denotes the flow of $Y \in E^{uu}$. It is enough to show that $\alpha_s^\sharp = 0$. Since $\mathbf{h}^{uu}_{p,q}(x) = \phi_{\zeta(x)}(x)$, for some C^1-function ζ, it follows that

$$T_x\mathbf{h}^{uu}_{p,q}(w) = T_x\phi_{\zeta(x)}(w) + d\zeta(w)Y,$$

(Observe that $\zeta(x) = \mathbf{h}^{cs}_{p,x}(q)$; cf., §3-C.) By Step 3A and $\alpha(Y) = 0$, we obtain $\alpha_s(w) = 0$, for $w \in F_{\ell-1}$ and every p and q, where $s = \zeta(x)$. It follows that $\alpha_s \restriction_{F_{\ell-1}} = 0$ for all $s \in \mathbb{R}$, so

$$F_{\ell-1} \subset \operatorname{Ker}(\alpha_s^\sharp).$$

Since $\alpha_s^\sharp$ is a 1-form on E^{ss}, its kernel is either $F_{\ell-1}$ or it is all of E^{ss}. Suppose that for some $s \neq 0$ the former holds at some point $x \in M$. Since

$$f_t^* \alpha_s = \alpha_{se^{-t}},$$

the kernel of $\alpha^\sharp_{se^{-t}}$ must equal $F_{\ell-1}$ for *all* $t \in \mathbb{R}$. In particular, this implies that $\alpha_s^\sharp$ can be viewed as a volume form for E_ℓ.

Furthermore, since they have the same kernel, $\alpha_s^\sharp$ and $\alpha^\sharp_{se^{-t}}$ are scalar multiples of each other. Let us take $s = 1$. Then there exists a function $t \mapsto k(t)$ such that for all t,

$$\alpha^\sharp_{e^{-t}} = k(t)\alpha_1^\sharp. \tag{4.6}$$

Therefore, $f_t^*\alpha_1^\sharp = k(t)\alpha_1^\sharp$, so $k(t)$ is the determinant of $Tf_t \restriction_{E_\ell}$ relative to the volume form $\alpha_1^\sharp$.

On the other hand, for small $r > 0$, in any set of local coordinates we have $\|T\phi_r - I\| \leq e^{\text{Lip}(Y)r} - 1$, where $\text{Lip}(Y)$ is the Lipschitz constant of Y. Thus

$$\begin{aligned}
\|\alpha^\sharp_{e^{-t}}\|_{C^0} &= \|\alpha^\sharp_{e^{-t}} - \alpha^\sharp_0\|_{C^0} \\
&\leq \|\phi^*_{e^{-t}}\alpha - \alpha\|_{C^0} \\
&\leq \|\alpha\|_{C^0} \left(e^{\text{Lip}(Y)e^{-t}} - 1 \right).
\end{aligned}$$

Since $(e^r - 1)/r \to 1$, as $r \to 0$, it follows that as $t \to +\infty$, $e^{\text{Lip}(Y)e^{-t}} - 1$ is asymptotically equivalent to $\text{Lip}(Y)e^{-t}$. Therefore, the left hand side of (4.6) converges to zero as e^{-t}. However, since $k(t)$ is the determinant of $Tf_t \restriction_{E_\ell}$ relative to the volume form $\alpha_1^\sharp$, and $\dim E_\ell = 1$, $k(t)$ is asymptotically equivalent to $\|Tf_t \restriction_{E_\ell}\|$, which goes to zero as $e^{-\chi_\ell t}$. This is a contradiction, since $\chi_\ell < 1$. Thus

$\alpha_s^\sharp = 0$, for all s.

Step 5. In summary, we have shown that $T\mathbf{h}_{p,q}^{uu}(E^{ss}) = E^{ss}$, for all $p \in M$ and $q \in W_{\text{loc}}^{uu}(p)$. This proves joint integrability of W^{ss} and W^{uu} and the existence of a smooth constant first-return time global cross section Σ for the flow of X. Vector fields X and X_2 have the same orbits, so Σ is a cross section for X_2. Therefore, the flow of X_2 is topologically equivalent to a suspension of a linear toral automorphism. Since $X_2 \in \mathscr{U}$, the same is true for the flow of X_0. To show that X_0 also admits a smooth global cross section, we use Proposition 1.1 from Ghys [13]: if Φ is a transitive codimension one Anosov flow, then Φ admits a global cross section if and only if no periodic orbit of Φ is homologous to zero. Since the flow of X_2 has this property, so does the topologically equivalent flow of X_0. This completes the proof.

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