Name: Granwyth Hulatberi

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

Explain your work
1. (25 points) Consider the differential equation
\[
\frac{dy}{dt} = y^2 - 2ty + t^2 - 3.
\]

(a) Perform the change of variables defined by \(u = y - t\).

(b) Find the equilibria of the \(u\)-equation.

(c) Sketch the phase line and graphs of a few representative solutions \(u(t)\) of the \(u\)-equation.

(d) Using (c), sketch the graphs of a few representative solutions \(y(t)\) of the original, \(y\)-equation.

Solution: (a) Since \(y^2 - 2ty + t^2 - 3 = (y - t)^2 - 3 = u^2 - 3\), we have:
\[
\frac{du}{dt} = \frac{dy}{dt} - 1 = (u^2 - 3) - 1 = u^2 - 4.
\]

(b) The equilibria of \(\frac{du}{dt} = u^2 - 4\) are \(u_- = -2\) and \(u_+ = 2\).

(c) Since \(u^2 - 4 < 0\) for \(-2 < u < 2\) and \(u^2 - 4 > 0\) for \(|u| > 2\), \(u_- = -2\) is a sink and \(u_+ = 2\) is a source. See Figure 1. For each solution \(u(t)\) of the \(u\)-equation, we obtain a solution \(y(t) = u(t) + t\) of the \(y\)-equation. Therefore, the graphs of solutions \(y(t)\) are obtained by shearing the \((t, u)\)-plane counterclockwise by 45 degrees.

![Figure 1: Problem 1.(c) and (d).](image-url)
2. (25 points) Consider the second-order equation

\[ \frac{d^2y}{dt^2} - y = 0. \]

(a) Convert it into a first-order system, \( \frac{dy}{dt} = AY. \)

(b) Find the eigenvalues and eigenvectors of \( A. \)

(c) Compute the general solution of the system.

(d) Sketch the phase portrait and identify the type of equilibrium at \((0, 0).\)

Solution: (a) Introducing a new variable \( v = \frac{dy}{dt}, \) we obtain the system

\[ \frac{dY}{dt} = AY, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y \\ v \end{bmatrix}. \]

(b) The characteristic equation is \( \det(A - \lambda I) = \lambda^2 - 1, \) so the eigenvalues of \( A \) are \( \lambda_1 = -1 \) and \( \lambda_2 = 1. \)

Let \( V_1 = (x_1, y_1) \) be an eigenvector corresponding to \( \lambda_1. \) Then \( (A - \lambda_1 I)V_1 = (A + I)V_1 = 0, \) which is equivalent to the system

\[
\begin{align*}
x_1 + y_1 &= 0 \\
x_1 + y_1 &= 0.
\end{align*}
\]

We can take \( x_1 = 1 \) and \( y_1 = -1, \) so \( V_1 = (1, -1). \) The matrix is symmetric, so the other eigenvector, \( V_2, \) is perpendicular to \( V_1. \) Therefore, we can take \( V_2 = (1, 1). \)

(c) The general solution of the system is

\[ Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2 = k_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

(d) Since the eigenvalues have different signs, the equilibrium is a saddle. See Fig. 2.
3. (25 points) For the planar linear system

\[ \frac{dY}{dt} = BY, \quad \text{where} \quad B = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix} : \]

(a) Find all straight-line solutions.

(b) Compute the particular solution satisfying the initial condition \( Y(0) = (0, -3) \).

(c) Sketch the phase portrait and identify the type of equilibrium at \((0,0)\).

Solution: (a) First we have to find the eigenvalue and eigenvectors of \( B \). The characteristic equation is

\[ \det(B - \lambda I) = \lambda^2 + 7\lambda + 10 = (\lambda + 2)(\lambda + 5), \]

so the eigenvalues are \( \lambda_1 = -2 \) and \( \lambda_2 = -5 \). Let \( V_i = (x_i, y_i) \) be the eigenvector corresponding to \( \lambda_i \), for \( i = 1, 2 \). Then \( (A - \lambda_i I)V_i = 0 \). For \( i = 1 \), we obtain the system

\[ \begin{align*}
-2x_1 - 2y_1 &= 0 \\
-x_1 - y_1 &= 0.
\end{align*} \]

Since \( y_1 = -x_1 \), we can take \( V_1 = (1, -1) \). For \( i = 2 \), we obtain the system

\[ \begin{align*}
x_2 - 2y_2 &= 0 \\
x_2 + 2y_2 &= 0.
\end{align*} \]

Since \( x_2 = 2y_2 \), we can take \( V_2 = (2, 1) \). Therefore, the straight line solutions are all scalar multiples of

\[ Y_1(t) = e^{\lambda_1 t}V_1 = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Y_2(t) = e^{\lambda_2 t}V_2 = e^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]
(c) The general solution is $Y(t) = k_1 Y_1(t) + k_2 Y_2(t)$. If $Y(0) = (0, -3)$, then $(0, -3) = k_1 Y_1(0) + k_2 Y_2(0)$, which gives us the following system:

\[
\begin{aligned}
k_1 + 2k_2 &= 0 \\
-k_1 + k_2 &= -3.
\end{aligned}
\]

To solve, we can add the two equations and obtain $k_2 = -1$ and $k_1 = 2$. Therefore, the desired particular solution is

\[
Y(t) = 2e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - 2e^{-5t} \\ -2e^{-2t} - e^{-5t} \end{bmatrix}.
\]

(d) The equilibrium is a sink, since both eigenvalues are negative (Fig. 3).

Figure 3: Problem 3. (d).
4. **(25 points)** Consider the nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= y - x^3 + x \\
\frac{dy}{dt} &= -x.
\end{align*}
\]

**(a)** Find the equilibria.

**(b)** Let \( \mathbf{F}(x, y) \) be the corresponding vector field. Compute \( \mathbf{F}(1, 0) \) and \( \mathbf{F}(0, 1) \).

**(c)** Is \( \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -1 \\ t \end{bmatrix} \) a solution? Explain why.

**Solution:**

(a) The equilibria are solutions of the system

\[
\begin{align*}
y - x^3 + x &= 0 \\
-x &= 0.
\end{align*}
\]

Therefore, \( x = 0 \) and \( y = 0 \). The only equilibrium is \((0, 0)\).

(b) We have

\[
\mathbf{F}(x, y) = \begin{bmatrix} y - x^3 + x \\ -x \end{bmatrix},
\]

so

\[
\mathbf{F}(1, 0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{F}(0, 1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

(c) Let \( Y(t) \) denote the given function. Then

\[
\frac{dY}{dt} = \frac{d}{dt} \begin{bmatrix} -1 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

whereas

\[
\mathbf{F}(Y(t)) = \mathbf{F}(-1, t) = \begin{bmatrix} t \\ 1 \end{bmatrix}.
\]

Since \( \frac{dY}{dt} \neq \mathbf{F}(Y(t)) \), \( Y(t) \) is **not** a solution.