THE FLOW OF A DIFFERENTIAL EQUATION

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Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ vector field. Then for each initial condition $X_0 \in \mathbb{R}^n$, the ODE $X' = F(X)$ has a unique solution, which we denote by $X(t)$. Thus $X(0) = X_0$ and $X'(t) = F(X(t))$. The flow

$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

of $X' = F(X)$ (or of $F$) is defined by

$$\varphi(t, X_0) = X(t).$$

Therefore, the defining properties of $\varphi$ are:

$$\varphi(0, X_0) = X_0, \quad \text{and} \quad \frac{d}{dt} \varphi(t, X_0) = F(\varphi(t, X_0)),$$

for all $t$. The time-$t$ map of the flow is the map

$$\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$\varphi_t(X_0) = \varphi(t, X_0).$$

We often abuse the terminology and call the collection \{\varphi_t\} of time-$t$ maps the flow.

So, for any initial state $X_0$ of the system, the time-$t$ map tells us the state of the system after $t$ units of time. $\varphi_t$ is a certain transformation of the phase space, $\mathbb{R}^n$. In fact, because of uniqueness of solutions, we know that

$$\varphi_0 = \text{identity} \quad \text{and} \quad \varphi_{s+t} = \varphi_s \circ \varphi_t,$$

where $\circ$ denotes composition of maps. This implies that $\varphi_t$ is always invertible and $(\varphi_t)^{-1} = \varphi_{-t}$. Moreover, we know (though we didn’t prove) that for every $t$, $\varphi_t$ is $C^1$ (actually, as smooth as $F$). So each time-$t$ map is a diffeomorphism of the phase space.

The flow notation is a convenient way of representing solutions of an ODE, but it’s also more than that. Namely, when we write $X(t)$, we are only paying attention to how one particular solution depends on time. When writing $\varphi_t(X_0)$, we care about the dependence of every solution on the initial condition, thus adopting a more global point of view. This way $X_0$ becomes a variable and since the subscript 0 in $X_0$ suggests that $X_0$ is somehow “fixed”, without fear of confusion, we rename the initial condition $X_0$ into $X$ and write $\varphi_t(X)$ for the solution that starts at $X$ at time $t = 0$.

**Example 1.** If $F$ is linear, $F(X) = AX$, where $A$ is an $n \times n$ matrix, then the flow is

$$\varphi_t(X) = \exp(tA)X.$$

Therefore, each time-$t$ map $\varphi_t$, is a linear map. The flow of a linear vector field is itself linear. For instance, if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

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then $\varphi_t$ is the clockwise rotation by $t$ radians.

**Example 2.** If $F(x, y) = (x + y^2, -y)$, then the flow $F$ (as shown in class) is

$$\varphi_t(x, y) = \left( (x + \frac{1}{3} y^2)e^t - \frac{1}{3} y^2 e^{-2t}, ye^{-t} \right).$$

Note how the flow of a nonlinear vector field is nonlinear. To see what happens to a particular point as $t$ varies, take $(x, y) = (-1, \sqrt{3})$; then

$$\varphi_t(-1, \sqrt{3}) = (-e^{-2t}, \sqrt{3}e^{-t}).$$

This is the solution that starts at $(-1, \sqrt{3})$. The phase portrait is given in Figure 1. The origin is a saddle.

![Figure 1. The phase portrait.](image-url)

What does the diffeomorphism $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ (for a fixed $t$) do to various types of sets in the plane? For instance, if

$$S = \{(x, y) : 0 \leq x, y \leq 1\},$$

i.e., $S$ is the unit square, what is the image $\varphi_t(S)$ of $S$ under $\varphi_t$? To find out, let’s first see what $\varphi_t$ does to horizontal and vertical lines. If $H$ is the horizontal line $y = c$ (constant), then for any $(x, c) \in H$, we have

$$\varphi_t(x, c) = \left( (x + \frac{1}{3} c^2)e^t - \frac{1}{3} c^2 e^{-2t}, ce^{-t} \right).$$

Observe that the $y$-coordinate is constant, since it doesn’t depend on $x$. (Remember that $t$ is fixed.) Therefore, $\varphi_t$ takes horizontal lines to horizontal lines.

Let $V$ be the vertical line $x = c$. Then for any $(c, y) \in V$, we have

$$\varphi_t(c, y) = \left( (c + \frac{1}{3} y^2)e^t - \frac{1}{3} y^2 e^{-2t}, ye^{-t} \right).$$

As $y$ varies, what type of curve does this point traverse? To answer this question, set $\varphi_t(c, y) = (u, v)$ and express $u$ in terms of $v$. After a little bit of work, we obtain

$$u = ce^t + \frac{1}{3} (e^{3t} - 1)v^2,$$
which defines a parabola. Therefore, \( \varphi_t \) takes vertical lines to parabolas. This means that \( \varphi_t(S) \) is the set bounded by two horizontal line segments and two parabolic segments as in Figure 2.

**Figure 2.** The image of the unit square \( S \) under the time-\( t \) map of the flow.

**Bonus.** What is the area of \( \varphi_t(S) \)? There is a result in geometry that says that if the divergence \( \text{div} \, F \) is positive, then, for \( t > 0 \), \( \varphi_t \) expands area, if \( \text{div} \, F < 0 \), it shrinks it, and if \( \text{div} \, F = 0 \), then \( \varphi_t \) is area preserving. Recall that the divergence of \( F = (f, g) \) is defined by

\[
\text{div} \, F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.
\]

Since in our case \( f(x, y) = x + y^2 \) and \( g(x, y) = -y \), the divergence is zero, so \( \varphi_t \) preserves area. Therefore,

\[
\text{area}(\varphi_t(S)) = \text{area}(S) = 1.
\]

**Remark.** (a) Recall that the existence and uniqueness theorem for ODEs guarantees that solutions are defined only for \( t \) close to 0. Therefore, our flow \( \varphi(t, X) \) is defined only for \( t \) in some neighborhood \( J \subset \mathbb{R} \) of zero. This neighborhood in general depends on \( X \), so we can write \( J = J(X) \). The flow should therefore be called the local flow, to indicate that solutions are only defined locally (in \( t \)). It would also be more correct to say that \( \varphi \) is defined on the set

\[
\Omega = \{(t, X) \in \mathbb{R} \times \mathbb{R}^n : t \in J(X)\};
\]

*not on all of \( \mathbb{R} \times \mathbb{R}^n \).

(b) Given a (smooth) collection of maps \( \varphi_t : \mathbb{R}^n \to \mathbb{R}^n \) satisfying \( \varphi_0 = \text{identity} \) and \( \varphi_{s+t} = \varphi_s \circ \varphi_t \), we can always recover the vector field \( F \) so that \( \varphi_t \) is the flow of \( F \). Just differentiate with respect to \( t \):

\[
F(X) = \left. \frac{d}{dt} \varphi_t(X) \right|_{t=0}.
\]