Name: Granwyth Hulatberi

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Explain your work
1. (25 points) Let \( \sigma : U \to \mathbb{R}^3 \) be defined by
\[
\sigma(u, v) = (u \cosh v, u \sinh v, u^2),
\]
where \( U = \{(u, v) \in \mathbb{R}^2 : u > 0\} \). Show that \( \sigma \) is a regular surface patch for the hyperbolic paraboloid
\[
S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}.
\]

**Proof:** If we write \( \sigma(u, v) = (x, y, z) \), then
\[
x^2 - y^2 = u^2(\cosh^2 v - \sinh^2 v) = u^2 = z,
\]
so \( \sigma(u, v) \in S \), for all \((u, v) \in U \). Let \( V = \sigma(U) \).
If \( \sigma(u_1, v_1) = \sigma(u_2, v_2) \), then
\[
u_1 \cosh v_1 = u_2 \cosh v_2, \quad u_1 \sinh v_1 = u_2 \sinh v_2, \quad u_1^2 = u_2^2.
\]
Thus \( u_1 = \pm u_2 \). But \( u_1, u_2 > 0 \), so \( u_1 = u_2 \). Since \( \sinh \) is 1–1, it follows that \( v_1 = v_2 \).
Therefore, \( \sigma \) is 1–1.
Since its components are elementary functions, \( \sigma \) is smooth (hence continuous). To find the inverse of \( \sigma \) on \( V \), we have to solve the equations
\[
x = u \cosh v, \quad y = u \sinh v, \quad z = u^2.
\]
We obtain
\[
u = \sqrt{z}, \quad v = \tanh^{-1} \left( \frac{y}{x} \right).
\]
Therefore, the map \((x, y, z) \mapsto (u, v)\) from \( V \) to \( U \) is continuous. This proves that \( \sigma : U \to V \) is a homeomorphism. It remains to verify that \( \sigma \) is regular. We have
\[
\sigma_u = (\cosh v, \sinh v, 2u), \quad \sigma_v = (u \sinh v, u \cosh v, 0),
\]
so
\[
\sigma_u \times \sigma_v = \begin{vmatrix}
i & j & k \\
\cosh v & \sinh v & 2u \\
u \sinh v & u \cosh v & 0
\end{vmatrix} = (-2u^2 \cosh v, 2u^2 \sinh v, u),
\]
which is non-zero for all \((u, v) \in U \). This completes the proof.
2. (25 points) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function. Show that the graph \( G \) of \( f \) is diffeomorphic to a plane.

**Proof:** It was shown in one of the homework assignments that \( G \) is a smooth surface covered by a single surface patch

\[
\sigma(u, v) = (u, v, f(u, v)), \quad (u, v) \in \mathbb{R}^2.
\]

As a surface patch, \( \sigma \) is a homeomorphism onto \( G \) and it is smooth. Its inverse is a map from \( G \) to \( \mathbb{R}^2 \) given by \((x, y, z) \mapsto (x, y)\). While this map is smooth as a map from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \), since we are restricting it to \( G \), we need to show that its representation in the surface patch \( \sigma \) is smooth. But that representation is \((u, v) \mapsto (\sigma^{-1} \circ \sigma)(u, v) = (u, v)\), i.e., the identity map on \( \mathbb{R}^2 \), which is clearly smooth. Therefore, \( \sigma : \mathbb{R}^2 \to G \) is a diffeomorphism.
3. **(25 points)** Let \( f(x, y, z) = (x + y + z - 1)^2 \). For what values of \( c \) is the set \( S_c = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\} \) a smooth surface?

**Solution:** If \( c < 0 \), then \( S_c \) is clearly empty. If \( c = 0 \), then \( S_c \) is the plane \( x + y + z = 1 \), hence a smooth surface. Assume \( c > 0 \). We have
\[
 f_x = f_y = f_z = 2(x + y + z - 1),
\]
so
\[
 \nabla f(x, y, z) = (f_x, f_x, f_x),
\]
which is \( \neq 0 \) iff \( f_x \neq 0 \). If \( p = (x, y, z) \in S_c \), then \( x + y + z = \pm \sqrt{c} + 1 \), so
\[
 f_x(p) = 2(x + y + z - 1) = \pm \sqrt{c} \neq 0,
\]
which implies that \( \nabla f(p) \neq 0 \). By a result proved in class (and based on the Implicit Function Theorem), \( S_c \) is a smooth surface. Therefore, \( S_c \) is empty for \( c < 0 \) and it is a smooth surface for all \( c \geq 0 \).

Alternatively, observe that for \( c > 0 \), \( S_c \) is the union of two parallel planes, hence a smooth surface.
4. **(25 points)** Show that the paraboloid

\[ S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\} \]

is orientable.

**Proof:** Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x, y) = x^2 + y^2 \). Since \( f \) is smooth, we know that its graph, namely, \( S \), is a smooth surface. It was shown in problem 2 that \( S \) is diffeomorphic to a plane. Every plane is an orientable surface. By a homework exercise (stating that if \( S \) is diffeomorphic to an orientable surface, then \( S \) is itself orientable), it follows that \( S \) is orientable.

Alternatively,

\[ \sigma(u, v) = (u, v, f(u, v)) \]

is a surface patch covering the whole surface and defining a smooth unit normal

\[ N = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}} \]

on the entire surface.