5.3, ex. 3: Let us inductively define a sequence \((x_n)\) in \(I = [a, b]\) as follows. Let \(x_1 = a\). Then, by assumption, there exists \(x_2 \in I\) such that \(|f(x_2)| \leq |f(x_1)|/2\). Applying the assumption again, we obtain \(x_3 \in I\) such that \(|f(x_3)| \leq |f(x_2)|/2\). Continuing in this way, we obtain a sequence \((x_n)\) in \(I\) satisfying

\[
|f(x_n)| \leq \frac{1}{2} |f(x_{n-1})| \leq \frac{1}{2^2} |f(x_{n-2})| \leq \cdots \leq \frac{1}{2^{n-1}} |f(x_1)|.
\]

Therefore, \(f(x_n) \to 0\), as \(n \to \infty\). Since \((x_n)\) is bounded, by Bolzano-Weierstraß it has a convergent subsequence \((x_{n_k})\). Call its limit \(x_*\). As \(a \leq x_n \leq b\), for all \(n \in \mathbb{N}\), it follows that \(x_* \in I\). Since \(f\) is continuous, we obtain

\[
f(x_*) = \lim_{k \to \infty} f(x_{n_k}) = 0. \quad \Box
\]

5.3, ex. 6: Let

\[g(x) = f(x) - f\left(x + \frac{1}{2}\right).\]

It suffices to show that \(g(c) = 0\), for some \(0 \leq c \leq 1/2\).

Since \(f\) is continuous, so is \(x \mapsto f\left(x + \frac{1}{2}\right)\), as the composition of two continuous functions. It follows that \(g\) is continuous on its domain \([0, \frac{1}{2}]\). Furthermore, \(g(0) = f(0) - f(1/2)\) and

\[g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0).\]

If \(g(0) = 0\), then \(f(0) = f(1/2)\), so we can take \(c = 0\). Otherwise, \(g(0) \neq 0\), so \(g(0)\) and \(g(1/2)\) are of opposite sign. By the Intermediate Value Theorem (IVT), it follows that there exists \(c \in (0, 1/2)\) such that \(g(c) = 0\). \(\Box\)

5.3, ex. 13: Since \(f(x) \to 0\), as \(x \to \pm \infty\), it follows that there exists \(K > 0\) such that \(|f(x)| < 1\) for all \(x\) such that \(|x| > K\). On the other hand, \(|f|\) is continuous, so it is bounded on the compact interval \([-K, K]\), say by \(L\). We obtain \(|f(x)| \leq \max\{1, L\}\), for all \(x \in \mathbb{R}\), so \(f\) is bounded on \(\mathbb{R}\), hence \(m = \inf_{\mathbb{R}} f\) and \(M = \sup_{\mathbb{R}} f\) are finite.

Since \(M = \sup f\), there exist a sequence \((x_n)\) such that \(f(x_n) \to M\), as \(n \to \infty\). Similarly, there exists a sequence \((y_n)\) such that \(f(y_n) \to m\). If \((x_n)\) is bounded, then by Bolzano-Weierstraß it has a convergent subsequence \(x_{n_k} \to c\). Thus, by continuity, \(f(x_{n_k}) \to f(c) = M\), which means that \(f\) attains its absolute maximum at \(c\). Similarly, if \((y_n)\) is bounded, then \(f\) attains its absolute minimum. If both \((x_n)\) and \((y_n)\) are unbounded, then they may not diverge to plus or minus infinity, but they do have to have subsequences \((x_{n_k})\) and \((y_{n_k})\) such that \(|x_{n_k}| \to \infty\) and \(|y_{n_k}| \to \infty\), as \(k \to \infty\). Since the limit of \(f\) at both \(\pm \infty\) equals zero, we obtain

\[M = \lim_{k \to \infty} f(x_{n_k}) = 0 \quad \text{and} \quad m = \lim_{k \to \infty} f(y_{n_k}) = 0.\]

If the maximum and minimum of \(f\) are both zero, then \(f(x) = 0\), for all \(x \in \mathbb{R}\), so \(f\) attains its absolute maximum and minimum (equal to zero) at every point.

To show that \(f\) does not have to attain both its absolute maximum and its absolute minimum, take

\[f(x) = \frac{1}{1 + x^2}. \quad \Box\]
5.3, ex. 17: Suppose $f$ is not constant. Then it takes at least two distinct values, say $u < v$. Since $f$ is continuous, by the Intermediate Value Theorem, every number in $(u, v)$ is a value of $f$. But every open interval $(u, v)$ contains both rational and irrational numbers, contradicting the assumption that $f$ takes only rational or only irrational values.

\[\square\]

5.4, ex. 2: Suppose $x, y \in A = [1, \infty)$. Then

\[
|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{x + y}{x^2y^2}|x - y| = \left( \frac{1}{xy^2} + \frac{1}{x^2y} \right)|x - y| \leq \left( \frac{1}{1} + \frac{1}{1} \right)|x - y| = 2|x - y|.
\]

Therefore, $f$ is Lipschitz on $A$, and as such uniformly continuous. Let us use the sequential criterion to show that $f$ is not uniformly continuous on $B = (0, \infty)$. Define $x_n = 1/\sqrt{n} + 1$ and $y_n = 1/\sqrt{n}$. Then $x_n - y_n \to 0$, as $n \to \infty$, whereas

\[f(x_n) - f(y_n) = (n + 1) - n = 1 \neq 0,
\]

proving the claim.

\[\square\]

5.4, ex. 8: Suppose that both $f$ and $g$ are uniformly continuous on $\mathbb{R}$. Let us use the Sequential Criterion for uniform continuity to show that $f \circ g$ is uniformly continuous on $\mathbb{R}$. Suppose that $(x_n)$ and $(y_n)$ are sequences of real numbers such that $x_n - y_n \to 0$, as $n \to \infty$. Since $g$ is uniformly continuous, it follows by the Sequential Criterion that $g(x_n) - g(y_n) \to 0$, as $n \to \infty$. Let $u_n = g(x_n)$ and $v_n = g(y_n)$. Since $f$ is uniformly continuous, by the Sequential Criterion we obtain $f(u_n) - f(v_n) \to 0$. But $f(u_n) = (f \circ g)(x_n)$ and $f(v_n) = (f \circ g)(y_n)$. Therefore, again by the Sequential Criterion, it follows that $f \circ g$ is uniformly continuous on $\mathbb{R}$.

\[\square\]

5.6, ex. 8: Suppose the contrary, i.e., $f^{-1}(y) \geq g^{-1}(y)$, for some $y \in f(I) \cap g(I)$. Let $x_1 = f^{-1}(y)$ and $x_2 = g^{-1}(y)$. Since $f$ is increasing, $x_1 \geq x_2$ implies $f(x_1) \geq f(x_2)$. On the other hand, the assumption $f > g$ on $I$ implies that $f(x_2) > g(x_2)$. Therefore, $f(x_1) > g(x_2)$. But $f(x_1) = f(f^{-1}(y)) = y$ and $g(x_2) = g(g^{-1}(y)) = y$, so $y > y$ – an impossibility. Therefore, $f^{-1}(y) < g^{-1}(y)$, for all $y \in f(I) \cap g(I)$.

\[\square\]

Remark: Recall that the graph of $f^{-1}$ is obtained by reflecting the graph of $f$ relative to the line $L : y = x$, and similarly for the graph of $g$. Geometrically, $f > g$ means that the graph of $f$ is above the graph of $g$. Applying the reflection relative to $L$, we obtain that the graph of $f^{-1}$ is below the graph of $g^{-1}$, i.e., $f^{-1} < g^{-1}$, wherever both function are defined.

5.6, ex. 9: Suppose that $f(u) = f(v)$, for some $u, v \in I$. If $u, v \in \mathbb{Q}$, then by definition of $f$, we have $u = v$. If $u, v \notin \mathbb{Q}$, then $1 - u = 1 - v$, which implies $u = v$. If $u \notin \mathbb{Q}$, but $v \in \mathbb{Q}$, then $u = 1 - v$, which is impossible, because $1 - v \in \mathbb{Q}$, and a number cannot be both rational and irrational. Therefore, $f$ is injective.
Let \( g = f \circ f \). If \( x \in \mathbb{Q} \), then clearly \( g(x) = x \). If \( x \not\in \mathbb{Q} \), then \( f(x) = 1 - x \not\in \mathbb{Q} \), so \( g(x) = f(1 - x) = 1 - (1 - x) = x \). In either case, \( g(x) = x \), i.e., \( f \circ f \) is the identity function.

In other words, \( f^{-1} = f \).

Suppose \( a \in I \) is arbitrary and assume \( x_n \to a \), as \( n \to \infty \). For rational \( x_n \), we have \( f(x_n) = x_n \to a \). For irrational \( x_n \), \( f(x_n) = 1 - x_n \to 1 - a \). Therefore, \( f \) has a limit at \( a \) if and only if \( a = 1 - a \), i.e., iff \( a = 1/2 \), and in that case the limit equals 1/2. Since \( f(1/2) = 1/2 \), it follows that \( f \) is continuous at 1/2. It is discontinuous elsewhere since it does not have a limit except at 1/2. \( \square \)

5.6, ex. 10: Suppose that \( f : [a, b] \to \mathbb{R} \) attains its absolute maximum at an interior point \( c \) of \( I = [a, b] \). Assume that \( f \) is injective on \( I \). If \( f \) is increasing, then \( f(c) < f(b) \). If \( f \) is decreasing, then \( f(a) > f(c) \). In either case, \( f(c) \) is not the absolute maximum of \( f \), contradicting our assumption. Therefore, \( f \) is not injective on \( I \). The case when \( f \) attains an absolute minimum at an interior point of \( I \) is handled similarly. \( \square \)

Remark: In the proof we used the fact that if \( f \) is 1–1 and continuous, then \( f \) is strictly monotonic. Here’s an alternative proof, which doesn’t use this fact. Suppose that \( f \) attains its absolute maximum at \( c \in (a, b) \). Then \( f(c) \geq f(x) \), for all \( x \in [a, b] \). If \( f(a) = f(c) \) or \( f(b) = f(c) \), we are done. Otherwise, \( f(a) < f(c) \) and \( f(b) < f(c) \). Let \( \lambda \) be an arbitrary number such that \( \max\{f(a), f(b)\} < \lambda < f(c) \). Then by the IVT there exist \( x_1 \in (a, c) \) such that \( f(x_1) = \lambda \). Similarly, there exists \( x_2 \in (c, b) \) such that \( f(x_2) = \lambda \). Since \( x_1 < x_2 \), it follows that \( f \) is not injective. \( \square \)

5.6, ex. 12: Assume \( x \in (0, 1) \). We claim that \( f(0) < f(x) < f(1) \). If \( f(x) < f(0) \) (note that \( f(x) \) must be different from \( f(0) \) since \( f \) does not take on any value twice), then by the IVT there exists \( c \in (x, 1) \), such that \( f(c) = f(0) \), which is impossible. Therefore, \( f(0) < f(x) \). If \( f(x) > f(1) \), then, again by the IVT, there exists \( c \in (0, x) \) such that \( f(c) = f(1) \), which is also impossible. Therefore, \( f(0) < f(x) < f(1) \). Let \( y \in (x, 1) \) be arbitrary. In a completely analogous way, we can show \( f(x) < f(y) < f(1) \). But this means that \( f \) is strictly increasing. \( \square \)