Today in class I mentioned that an inductively defined sequence can be viewed as a dynamical system. Here’s what I meant.

**Definition.** A *discrete dynamical system* (or *dynamical system* for short) on $\mathbb{R}$ is a continuous map $f : \mathbb{R} \to \mathbb{R}$. The *orbit* of a point $x_0 \in \mathbb{R}$ is the sequence $(x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots)$.

Here, $f^n = f \circ f \circ \cdots \circ f$, $f$ composed with itself $n$ times. In other words, the orbit of $x_0$ is the sequence defined inductively by

$$x_{n+1} = f(x_n),$$

with initial value $x_0$. The goal of the theory of dynamical systems is to understand the behavior of orbits of different points. For instance, *do they converge to some limit? How does that limit depend on the initial point, $x_0$?*

Suppose that an orbit $(x_n)$ does converge, say to $x_*$. Then (using the assumption that $f$ is continuous)

$$x_* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) = f(x_*).$$

That is to say, $x_*$ is a *fixed point* of $f$: if we apply $f$ to $x_*$, $x_*$ stays stationary (that’s why it’s sometimes called a *stationary point* or an *equilibrium*). In particular, this means that if $f$ has no fixed points, then $(x_n)$ can’t converge.

Let’s say that $f$ has fixed points. How do we know if for some choice of $x_0$, its orbit is going to converge to one of the fixed points or not? To answer this question, we need the Mean Value Theorem (MVT) which we will learn closer to the end of semester. The MVT yields the following result:

**Theorem.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f(x_*) = x_*$. If $|f'(x_*)| < 1$, then $x_*$ is an *attracting fixed point* of $f$, i.e., there exists an open interval $U$ containing $x_*$ such that for every $x_0 \in U$, the orbit of $x_0$ converges to $x_*$. 

**Example.** Let $f(x) = \lambda x(1 - x)$, where $\lambda$ is some in the interval $(1, 3)$. (If you want a more concrete example, take a concrete value such as $\lambda = 3/4$.) This $f$ is called the *logistic map.* Its graph is given in Fig. 1. To find the fixed points of $f$ we need to solve the equation $f(x) = x$. We get two solutions: $p_0 = 0$ and $p_1 = (\lambda - 1)/\lambda$. Differentiating, we obtain $f'(x) = \lambda - 2\lambda x$, so

$$f'(p_0) = \lambda \quad \text{and} \quad f'(p_1) = 2 - \lambda.$$ 

Since $\lambda \in (1, 3)$, it follows that $|f'(p_1)| = |2 - \lambda| < 1$, so $p_1$ is an attracting fixed point. Similarly $p_0$ is a *repelling* fixed point in the sense that if $x_0$ is close to $p_0$, then its orbit diverges from $p_0$.

If you are wondering what this is all for, consider the following story. In the 1960’s, a Berkeley biologist studied how the population of a certain type of flies changes from day to 

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day as the flies, kept in a closed environment (i.e., a box) are fed different amounts of food. He found that if given a sufficiently large amount of food, the population varies in an extremely unpredictable way. He also found that if $x_0$ is the initial population (normalized to be between 0 and 1), then the population size $x_n$ after $n$ days can be modeled as $x_{n+1} = f(x_n)$, where $\lambda$ is related to the amount of food. In particular, if $\lambda > 4$, then for most $x_0$, the sequence $(x_n)$ has a very complicated behavior – it jumps around without converging to a single value. We say that this dynamical system is chaotic.

If you think this is interesting and would like to learn more, you may enjoy Math 134: Dynamical Systems, which I’ll most likely teach in the Fall of 2007.