LIMSUP AND LIMINF

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Here is a brief exposition of some of the concepts I mentioned in class today.

**Definition.** A number $a \in \mathbb{R}$ is an accumulation point of a sequence $x = (x_n)$ if there exists a subsequence of $x$ whose limit is $a$. The set of all accumulation points of $x$ is denoted by $A(x)$.

In this terminology, the Bolzano-Weiestrass theorem can be formulated as follows:

**Theorem** (Bolzano-Weierstrass). Every bounded sequence has an accumulation point.

**Example 1.** If $x_n = (-1)^n$, then $-1$ and $1$ are accumulation points of $x$. If $a \not\in \{-1, 1\}$, then it is not hard to see that $a$ cannot be an accumulation point of $x$. Therefore, $A(x) = \{-1, 1\}$.

**Example 2.** Let a sequence $y = (y_n)$ be defined by

$$y_n = \begin{cases} \frac{1}{n} & \text{if } n = 3k, \text{ for some } k \in \mathbb{N} \\ 1 - 2^{-n} & \text{if } n = 3k + 1, \text{ for some } k \in \mathbb{N} \\ (1 + \frac{1}{n})^n & \text{if } n = 3k + 2, \text{ for some } k \in \mathbb{N}. \end{cases}$$

Since $y_{3k} \to 0$, $y_{3k+1} \to 1$ and $y_{3k+2} \to e$, as $k \to \infty$, we obtain that $A(y) = \{0, 1, e\}$.

**Example 3.** Since the set $S = \mathbb{Q} \cap [0, 1]$ is denumerable, there exists a bijection $r : \mathbb{N} \to S$. Note that $r = (r_n)$ is a sequence, with $r_n = r(n)$. Therefore, we can write $S = \{r_1, r_2, \ldots\}$.

In other words, $r$ gives one way of counting the rationals in $[0, 1]$. Since $0 \leq r \leq 1$, it follows that $A(r) \subset [0, 1]$. We claim that $[0, 1]$ is also contained in $A(r)$. To prove that, let $x \in [0, 1]$ be arbitrary. For each $k \in \mathbb{N}$, let $\varepsilon_k = 1/2^k$. Then, by the density of the rationals in the reals, for each $k$ there exists an element of $S$ in $(x - \varepsilon_k, x + \varepsilon_k)$. Since $S = \{r_1, r_2, \ldots\}$, we can call that element $r_{nk}$. This defines a subsequence $(r_{nk})$ of $r$ such that

$$|x - r_{nk}| < \varepsilon_k,$$

for all $k \in \mathbb{N}$. Since $\varepsilon_k = \frac{1}{2^k} \to 0$, as $k \to \infty$, it follows that $r_{nk} \to x$. Therefore, $x$ is an accumulation point of $r$. Since $x \in S$ was arbitrary, this proves that $A(r) = [0, 1]$.

We have a (countable) sequence which accumulates at every point of the uncountable set $[0, 1]$!

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If \( x = (x_n) \) is a bounded sequence, then there exist \( M > 0 \), such that \( -M \leq x_n \leq M \), for all \( n \in \mathbb{N} \). Therefore, all accumulation points of \( x \) have to be in \([-M, M]\). In other words, \( A(x) \) is a bounded set, hence it has a supremum and infimum. It turns out that \( A(x) \) actually contains its sup and inf. So the following definition makes sense:

**Definition.** \( \liminf_{n \to \infty} x_n = \min A(x) \) and \( \limsup_{n \to \infty} x_n = \max A(x) \).

**Example 4** (Examples 1, 2, and 3 revisited). We have

\[
\liminf_{n \to \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \to \infty} = 1,
\]
\[
\liminf_{n \to \infty} y_n = 0 \quad \text{and} \quad \limsup_{n \to \infty} y_n = e,
\]
\[
\liminf_{n \to \infty} r_n = 0 \quad \text{and} \quad \limsup_{n \to \infty} r_n = 1.
\]

Here’s an important characterization of liminf and limsup. For each \( n \in \mathbb{N} \), let

\[
T_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}.
\]

This is the \( n \)-tail of the sequence \((x_n)\). Since \((x_n)\) is bounded, so is each \( T_n \), so we can define

\[
a_n = \inf T_n \quad \text{and} \quad b_n = \sup T_n.
\]

Observe that \( T_{n+1} \subseteq T_n \), for each \( n \), so \((T_n)\) is a **decreasing** sequence of sets. Therefore, \( \sup T_{n+1} \leq \sup T_n \) and \( \inf T_{n+1} \geq \inf T_n \). That is, \((b_n)\) is decreasing and \((a_n)\) is increasing. Since they are both bounded (remember that \(|x_n| \leq M\), for all \( n \), so \(|a_n|, |b_n| \leq M\), for all \( n \)), it follows that they are both convergent. Here’s what can be said about their limits:

**Theorem.** (a) \( \lim_{n \to \infty} a_n = \liminf_{n \to \infty} x_n \).

(b) \( \lim_{n \to \infty} b_n = \limsup_{n \to \infty} x_n \).

The proof is left to the reader as an exercise. In fact, if you find a proof, I’ll add two extra percentage points to your overall score for the course.