## Math 131A, Fall 2006

### Midterm 1 Solutions

**September 27, 2006**

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**Score**

**Explain your work**
1. **(25 points)** Let $f : X \to Y$ be a function and suppose $A$ and $B$ are subsets of $Y$. Show that

$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B).$$

**Proof:** ($\subseteq$) Suppose $x \in f^{-1}(A \setminus B)$. Then $f(x) \in A \setminus B$. Therefore, $f(x) \in A$ and $f(x) \notin f(B)$. This implies that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. By definition of set difference, $x \in f(A) \setminus f(B)$. This proves that $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

($\supseteq$) Suppose $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, $f(x) \in A$ and $f(x) \notin B$. By definition of set difference, $f(x) \in A \setminus B$. Finally, by definition of the preimage, $x \in f^{-1}(A \setminus B)$. This proves that $f^{-1}(A \setminus B) \supseteq f^{-1}(A) \setminus f^{-1}(B)$ and completes the proof.
2. (25 points) Show that the set

\[ S = \left\{ \frac{m^2}{n^2} : m \in \mathbb{N}, n \in \mathbb{N} \right\} \]

is countable.

**Proof 1:** Define \( f : \mathbb{Q} \to S \) by

\[ f(x) = x^2. \]

We claim that \( f \) is a surjection. Let \( y \in S \) be arbitrary. Then \( y = \frac{m^2}{n^2} = (m/n)^2 \), for some \( m, n \in \mathbb{N} \). It follows that \( y = f(x) \), where \( x = m/n \in \mathbb{Q} \). This proves that \( f \) is onto.

Since \( \mathbb{Q} \) is countable and \( f \) maps \( \mathbb{Q} \) onto \( S \), \( S \) is countable.

**Proof 2:** A map \( g : \mathbb{N} \times \mathbb{N} \to S \) defined by \( g(m, n) = \frac{m^2}{n^2} \) is a surjection. Since \( \mathbb{N} \times \mathbb{N} \) is countable, so is \( S \).

**Proof 3:** Since \( S \subset \mathbb{Q} \) and \( \mathbb{Q} \) is countable, so is \( S \).
3. (25 points) Let $A = \{n^{-1} : n \in \mathbb{N}\}$. Compute $\inf A$.

**Proof:** First observe that $A = \{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots\}$.

We claim that the infimum of $A$ is zero. Since $n^{-1}$ is either equal to $n$ or $1/n$, 0 is a lower bound, so it suffices to show that it is the greatest lower bound. Suppose not, i.e., suppose that some $\epsilon > 0$ is also a lower bound for $A$. By the Archimedean property, there exists a natural number $n$ such that $n > 1/\epsilon$. We can assume $n$ is odd; otherwise, replace $n$ by $n + 1$. Since $n$ is odd, $n^{-1} = 1/n < \epsilon$. But $n^{-1} \in A$ – a contradiction! Therefore,

$$\inf A = 0.$$
4. (25 points) Prove that there exists no rational number whose cube equals 2.

Proof: Suppose the contrary, that is, there exists a rational number \( r \) such that \( r^3 = 2 \). Then \( r = m/n \), for some \( m \in \mathbb{Z} \), \( n \in \mathbb{N} \) and \( \gcd(m, n) = 1 \). Since \( r^3 = 2 \), we obtain

\[
m^3 = 2n^3.
\]

Therefore, \( m^3 \) is even. It follows that \( m \) must be even (otherwise, if \( m \) were odd, \( m^3 \) would be odd). Thus \( m = 2k \), for some integer \( k \). We obtain

\[
8k^3 = 2n^3, \quad \text{i.e.}, \quad 4k^3 = n^3.
\]

Therefore \( n^3 \) is even, so by the same argument as above, \( n \) is even. It follows that \( \gcd(m, n) \geq 2 \), contrary to our assumption. This completes the proof.