MATH 134, FALL 2007
HOMEWORK 1 SOLUTIONS

Ch. 1, ex. 5: $x' = ax + \sin x$

(a) When $a = 0$, then the equilibria are points $x_n = n\pi$, for all integers n. For even n, x_n is a source and for odd n, x_n is a sink.

(b) For each $-1 < a < 1$, the equation $x' = ax + \sin x$ has an odd number of equilibria. Zero is always a source and for each equilibrium $x_+ > 0$, there is a symmetric one $x_- = -x_+ < 0$. These points are solutions of the equation $ax + \sin x = 0$. (This is a transcedental equation, so there's no hope of computing its roots explicitly.) Recall that if x_0 is an equilibrium of $x' = f(x)$, then x_0 is a sink (source) if $f'(x_0) < 0$ ($f'(x_0) > 0$). Threfore, if x_+ is a sink, then so is x_-. Similarly, if x_+ is a source, then so is x_-. This follows from the fact that cos is an even function.

Ch. 1, ex. 8: Let $x(t)$ be an arbitrary solution to $x' = ax + f(t)$. Set $z(t) = x(t) - y(t)$. Then $z'(t) = az(t)$, for all t, so $z(t) = ce^{at}$. Therefore, $x(t) = y(t) + ce^{at}$.

Ch. 1, ex. 9: (a) Separating the variables, we obtain

$$\int_0^t \frac{x'(s)}{x(s)} ds = \int_0^t a(s)\, ds.$$

Therefore,

$$x(t) = x(0) \exp \left\{ \int_0^t a(s)\, ds \right\}.$$

(b) Let $y(t)$ be an arbitrary solution to $x' = a(t)x$. Set $z(t) = y(t) \exp \left\{ -\int_0^t a(s)\, ds \right\}$. Since $y'(t) = a(t)y(t)$, it is not hard to check that $z'(t) \equiv 0$. Therefore, $z(t) \equiv$ constant. This shows $y(t) = \text{constant} \cdot \exp \left\{ \int_0^t a(s)\, ds \right\}$, which is of the same form as the solution found in (a).

Ch. 1, ex. 11: (a) The general solution to $x' = x^2$ consists of

$$x(t) = -\frac{1}{t + C} \qquad \text{and} \qquad x(t) \equiv 0.$$

(b) The first solution in (a) is defined on the union of intervals $(-\infty, -C)$ and $(-C, \infty)$. The second one is defined for all t.

(c) Let $y(t) = \tan\left(\frac{\pi}{2} t\right)$. Then $y(t)$ is a solution to the differential equation $y' = \frac{\pi}{2}(1 + y^2)$ and it is defined for $-1 < t < 1$.

Ch. 1, ex. 12: (a) Separating the variables, we obtain that for every constant c,

$$x_c(t) = \left[\frac{2}{3}(t - c) \right]^{3/2}$$

is a solution to $x' = x^{1/3}$. To show that there are infinitely many solutions to the initial value problem $x' = x^{1/3}$, $x(0) = 0$, for any $c > 0$ define the following function:

$$y_c(t) = \begin{cases} 0 & \text{if } t \le c, \\ x_c(t) & \text{if } t > c. \end{cases}$$

Then $y_c(t)$ **is** a solution to $x' = x^{1/3}$ *and* $y_c(0) = 0$. There are infinitely many such solutions, one for each $c > 0$.

(b) Separation of variables yields the following solutions:

$$x(t) = Ct, \qquad \text{for } C \in \mathbb{R}.$$

Note that the ODE $x' = x/t$ makes sense only for $t \neq 0$, so the solutions are only defined for $t \neq 0$. However, if we rewrite the equation as $tx' = x$, then the above functions are solutions for **all** t. If $x_0 \neq 0$, then there are no solutions satisfying $x(0) = x_0$. If $x_0 = 0$, then there are infinitely many of them.

(c) We have the following solutions:

$$x(t) = Ce^{-1/t}, \qquad \text{for } C \in \mathbb{R},$$

defined for all $t \neq 0$. If we rewrite the ODE as $t^2x' = x$, then the equilibrium solution $x(t) \equiv 0$ is the unique one satisfying the initial condition $x(0) = 0$. For $C \neq 0$, the other solution cannot be extended continuously to $\mathbb{R}$, since the one-sided limits of $e^{-1/t}$ at zero are different: $e^{-1/t} \to 0$, as $t \to 0+$, but $e^{-1/t} \to \infty$, as $t \to 0-$.

Ch. 1, ex. 13: (a) Nothing can be said, in general. Examples: 1) Take $f(x) = x^2$ and $x_0 = 0$. Then all solutions, except the equilibrium one, increase, since $x'(t) = x(t)^2 > 0$. So 0 is neither a sink nor a source. 2) Take $f(x) = x^3$ and $x_0 = 0$. Then 0 is a source. 3) Take $f(x) = -x^3$ and $x_0 = 0$. Then 0 is a sink. In all examples, $f(x_0) = f'(x_0) = 0$.

(b) Now more can be said. If $f''(x_0) > 0$, then f has a local minimum at x_0, equal to $f(x_0) = 0$. Therefore, $f \geq 0$ in a neighborhood of x_0, so all solutions except for the equilibrium one, increase.

If $f''(x_0) < 0$, then f has a local maximum at x_0, equal to $f(x_0) = 0$. All solutions decrease, except for the equilibrium one.

(c) If $f'(x_0) = f''(x_0) = 0$, but $f'''(x_0) > 0$, then f' has a local minimum at x_0 equal to $f'(x_0) = 0$. Therefore, $f' \geq 0$ in a neighborhood U of x_0, so f is increasing on U. Therefore, $f(x) < 0$ for $x \in U$ with $x < x_0$ and $f(x) > 0$ for $x \in U$ with $x > x_0$. This means that x_0 is x_0 is a sink.

Similarly, if $f'(x_0) = f''(x_0) = 0$, but $f'''(x_0) < 0$, then x_0 is a source.

Ch. 1, ex. 14: By exercise 9, the general solution to $x' = p(t)x$ is

$$x(t) = x(0) \exp\left\{\int_0^t p(s)\, ds\right\}.$$

To examine when it is periodic with period T, we compute $x(t+T)$:

$$\begin{aligned}
x(t+T) &= x(0) \exp\left\{\int_0^{t+T} p(s)\, ds\right\} \\
&= x(0) \exp\left\{\int_0^{t} p(s)\, ds + \int_t^{t+T} p(s)\, ds\right\} \\
&= x(t) \exp\left\{\int_t^{t+T} p(s)\, ds\right\} \\
&\overset{s=r-T}{=} x(t) \exp\left\{\int_0^{T} p(r-T)\, dr\right\} \\
&= x(t) \exp\left\{\int_0^{T} p(r)\, dr\right\},
\end{aligned}$$

where in the last line we used the assumption that $p(t+T) \equiv p(t)$. Therefore, $x(t+T) \equiv x(t)$ if and only if

$$\exp\left\{\int_0^T p(r)\,dr\right\} = 1 \qquad \Leftrightarrow \qquad \int_0^T p(r)\,dr = 0.$$