MATH 134, FALL 2007
HOMEWORK 9 SOLUTIONS

Ch. 9, ex. 8(c): Since

$$\frac{\partial(x^2 - 2xy)}{\partial x} = -\frac{\partial(y^2 - 2xy)}{\partial y},$$

the system is Hamiltonian. Integrating, we obtain a Hamiltonian function

$$H(x, y) = x^2 y - xy^2.$$

The only equilibrium is the origin. The linearization matrix there is the zero matrix, which doesn't tell us anything about the original system. Since the system is Hamiltonian, the solutions are the level curves $H = c$. For $c = 0$ they are straight lines $x = 0$ or $y = 0$ or $x = y$; for $c \neq 0$, they are curves $xy^2 - x^2 y + c = 0$, or solving for y:

$$y = \frac{x^2 \pm \sqrt{x^4 - 4cx}}{2x} \qquad (x \neq 0).$$

The positive y-axis is a solution which diverges from the origin, so the origin is unstable. The phase portrait is sketched in Figure 1.

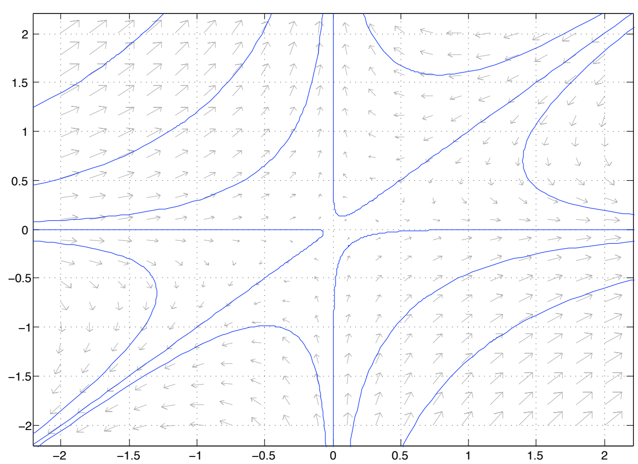

FIGURE 1. Phase portrait of the system in Ch. 9, ex. 8(c).

Ch. 9, ex. 16: Suppose $X(t)$ is a recurrent solution of a gradient system $X' = -\operatorname{grad} V(x)$, that is, $X(t_n) \to X(0)$, for some sequence $t_n \to \infty$. Without loss of generality, we can assume that $t_n > 0$, for all n, and that (t_n) is an increasing sequence. Then $V(X(0)) > V(X(t_n))$, for every $n \in \mathbb{N}$. On the other hand, $V(X(t_n)) \to V(X(0))$, as $n \to \infty$, since V is continuous. The sequence $V(X(t_n))$ is decreasing, so $V(X(t_n)) \geq V(X(0))$. But this is a contradiction since we already said that $V(X(0)) > V(X(t_n))$. Therefore, there can be no recurrent solutions in a gradient system. □

Chapter 13, ex. 1: (a) $F(x, y) = (-x^2, -2y^2) = -\operatorname{grad}(x^3/3 + 2y^3/3)$, so the force field is conservative.

(b) $F(x,y) = (x^2 - y^2, 2xy)$ fails the partial derivative test $(\partial(x^2-y^2)/\partial y \neq \partial(2xy)/\partial x)$ so it is not conservative.

(c) $F(x,y) = (x,0) = -\operatorname{grad}(x^2/2)$, hence it is conservative. □

Chapter 13, ex. 5: (⇒) Suppose $F = -\operatorname{grad} V$ is conservative. Then, by the chain rule and the fundamental theorem of calculus,

$$\begin{aligned} \text{work} &= \int_{s_0}^{s_1} F(Y(s)) \cdot Y'(s)\, ds \\ &= \int_{s_0}^{s_1} -\operatorname{grad} V(Y(s)) \cdot Y'(s)\, ds \\ &- \int_{s_0}^{s_1} \frac{d}{ds} V(Y(s))\, ds \\ &= V(Y(s_0)) - V(Y(s_1)) \\ &= V(X_0) - V(X_1). \end{aligned}$$

(⇐) Suppose the work is independent of the path. Define a function $V : \mathbb{R}^3 \to \mathbb{R}$ by

$$V(X) = -\int_0^1 F(Y(s)) \cdot Y'(s)\, ds,$$

where $Y : [0,1] \to \mathbb{R}^3$ is an arbitrary smooth path such that $Y(0) = \mathbf{0}$ and $Y(1) = X$. Then by assumption $V(X)$ does not depend on the choice of the path and by the fundamental theorem of calculus, we have

$$\operatorname{grad} V(X) = -F(X),$$

which means that F is conservative. □