THE FLOW OF A DIFFERENTIAL EQUATION

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Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field. Then for each initial condition $X_0 \in \mathbb{R}^n$, the ODE $X' = F(X)$ has a unique solution, which we denote by $X(t)$. Thus $X(0) = X_0$ and $X'(t) = F(X(t))$. The **flow**

$$\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

of $X' = F(X)$ (or of F) is defined by

$$\varphi(t, X_0) = X(t).$$

Therefore, the *defining properties* of φ are:

$$\varphi(0, X_0) = X_0, \qquad \text{and} \qquad \frac{d}{dt}\varphi(t, X_0) = F(\varphi(t, X_0)),$$

for all t. The **time-t map** of the flow is the map

$$\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$\varphi_t(X_0) = \varphi(t, X_0).$$

We often abuse the terminology and call the collection $\{\varphi_t\}$ of time-t maps the flow.

So, for any initial state X_0 of the system, *the time-t map tells us the state of the system after t units of time.* φ_t is a certain transformation of the phase space, $\mathbb{R}^n$. In fact, because of uniqueness of solutions, we know that

$$\varphi_0 = \text{identity} \qquad \text{and} \qquad \varphi_{s+t} = \varphi_s \circ \varphi_t,$$

where $\circ$ denotes composition of maps. This implies that φ_t is always invertible and $(\varphi_t)^{-1} = \varphi_{-t}$. Moreover, we know (though we didn't prove) that for every t, φ_t is C^1 (actually, as smooth as F). So each time-t map is a *diffeomorphism* of the phase space.

The flow notation is a convenient way of representing solutions of an ODE, but it's also more than that. Namely, when we write $X(t)$, we are only paying attention to how one particular solution depends on time. When writing $\varphi_t(X_0)$, we care about the dependence of *every solution on the initial condition*, thus adopting a more global point of view. This way X_0 becomes a variable and since the subscript 0 in X_0 suggests that X_0 is somehow "fixed", without fear of confusion, we rename the initial condition X_0 into X and write $\varphi_t(X)$ for the solution that starts at X at time $t = 0$.

Example 1. If F is linear, $F(X) = AX$, where A is an $n \times n$ matrix, then the flow is

$$\varphi_t(X) = \exp(tA)X.$$

Therefore, each time-t map φ_t, is a *linear map.* The flow of a linear vector field is itself linear. For instance, if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

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then φ_t is the clockwise rotation by t radians.

Example 2. If $F(x, y) = (x + y^2, -y)$, then the flow F (as shown in class) is

$$\varphi_t(x, y) = \left((x + \frac{1}{3}y^2)e^t - \frac{1}{3}y^2e^{-2t}, ye^{-t}\right).$$

Note how the flow of a nonlinear vector field is nonlinear. To see what happens to a particular point as t varies, take $(x, y) = (-1, \sqrt{3})$; then

$$\varphi_t(-1, \sqrt{3}) = (-e^{-2t}, \sqrt{3}e^{-t}).$$

This is the solution that starts at $(-1, \sqrt{3})$. The phase portrait is given in Figure 1. The origin is a saddle.

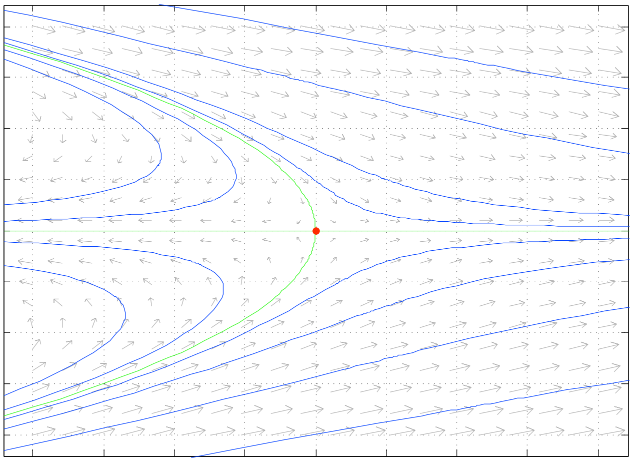

FIGURE 1. The phase portrait.

What does the diffeomorphism $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ (for a fixed t) do to various types of sets in the plane? For instance, if

$$S = \{(x, y) : 0 \leq x, y \leq 1\},$$

i.e., S is the unit square, what is the image $\varphi_t(S)$ of S under φ_t? To find out, let's first see what φ_t does to horizontal and vertical lines. If H is the horizontal line $y = c$ (constant), then for any $(x, c) \in H$, we have

$$\varphi_t(x, c) = \left((x + \frac{1}{3}c^2)e^t - \frac{1}{3}c^2e^{-2t}, ce^{-t}\right).$$

Observe that the y-coordinate is constant, since it doesn't depend on x. (Remember that t is fixed.) Therefore, φ_t takes horizontal lines to horizontal lines.

Let V be the vertical line $x = c$. Then for any $(c, y) \in V$, we have

$$\varphi_t(c, y) = \left((c + \frac{1}{3}y^2)e^t - \frac{1}{3}y^2e^{-2t}, ye^{-t}\right).$$

As y varies, what type of curve does this point traverse? To answer this question, set $\varphi_t(c, y) = (u, v)$ and express u in terms of v. After a little bit of work, we obtain

$$u = ce^t + \frac{1}{3}(e^{3t} - 1)v^2,$$

which defines a parabola. Therefore, φ_t takes vertical lines to parabolas. This means that $\varphi_t(S)$ is the set bounded by two horizontal line segments and two parabolic segments as in Figure 2.

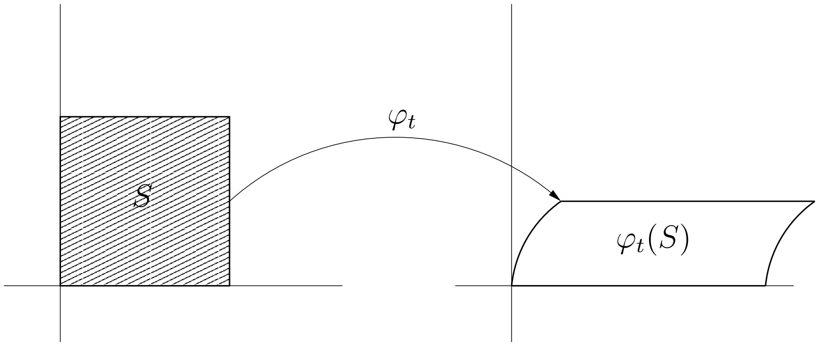

FIGURE 2. The image of the unit square S under the time-t map of the flow.

Bonus. What is the area of $\varphi_t(S)$? There is a result in geometry that says that if the *divergence* $\operatorname{div} F$ is positive, then, for $t > 0$, φ_t expands area, if $\operatorname{div} F < 0$, it shrinks it, and if $\operatorname{div} F = 0$, then φ_t is *area preserving.* Recall that the divergence of $F = (f, g)$ is defined by

$$\operatorname{div} F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

Since in our case $f(x, y) = x + y^2$ and $g(x, y) = -y$, the divergence is zero, so φ_t preserves area. Therefore,

$$\operatorname{area}(\varphi_t(S)) = \operatorname{area}(S) = 1.$$

Remark. (a) Recall that the existence and uniqueness theorem for ODEs guarantees that solutions are defined only for t close to 0. Therefore, our flow $\varphi(t, X)$ is defined only for t in some neighborhood $J \subset \mathbb{R}$ of zero. This neighborhood in general depends on X, so we can write $J = J(X)$. The flow should therefore be called the *local flow*, to indicate that solutions are only defined locally (in t). It would also be more correct to say that φ is defined on the set

$$\Omega = \{(t, X) \in \mathbb{R} \times \mathbb{R}^n : t \in J(X)\},$$

not on all of $\mathbb{R} \times \mathbb{R}^n$.

(b) Given a (smooth) collection of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\varphi_0 =$ identity and $\varphi_{s+t} = \varphi_s \circ \varphi_t$, we can always recover the vector field F so that φ_t is the flow of F. Just differentiate with respect to t:

$$F(X) = \left.\frac{d}{dt}\right|_{t=0} \varphi_t(X).$$

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