Name: Granwyth Hulatberi

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Explain your work
1. **(25 points)** Consider the differential equation

\[ x' = (x^2 - x - 2) \arctan x^2. \]

(a) Draw the phase line.

(b) Classify all equilibria.

(c) Sketch the *graph* of the solution satisfying the initial condition \( x(0) = 1 \).

**Solution:** (a) Since \( x^2 - x - 2 = (x + 1)(x - 2) \) and \( \arctan 0 = 0 \), the equilibria are \(-1, 0\), and \(2\). Observe that \( \arctan x^2 \geq 0 \), for all \( x \), so the sign of \( (x^2 - x - 2) \arctan x^2 \) equals the sign of \( x^2 - x - 2 \). This sign is negative on \((-1, 0)\) and \((0, 2)\), and positive otherwise. This means that solutions starting in \((-1, 0)\) and \((0, 2)\) decrease and all others (except for the equilibria) increase. Therefore, the phase line looks like this:

![Phase Line Diagram]

(b) It follows that \(-1\) is a sink, \(0\) is a node, and \(2\) is a source.

(c) Since \(0 < x(0) < 2\) and \( t \mapsto x(t) \) is decreasing, it follows that \( x(t) \to 0\), as \( t \to \infty \). Similarly, \( x(t) \to 2\), as \( t \to -\infty \). The graph of this solution looks approximately like this:

![Graph of Solution]
2. (25 points) Consider the planar linear system

\[ X' = AX, \quad \text{where} \quad A = \begin{bmatrix} -4 & -2 \\ -1 & -3 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \).
(b) Find a matrix \( T \) that puts \( A \) into canonical form. Write down the canonical form \( B \) of \( A \).
(c) Find the general solution of both \( X' = AX \) and \( Y' = BY \).
(d) Sketch the phase portraits of both systems. What type of equilibrium is this?

**Solution:** (a) The trace of \( A \) is \(-7\) and the determinant equals \(10\), so the eigenvalues of \( A \) are

\[ \lambda_{1,2} = \frac{\text{Trace}(A) \pm \sqrt{\text{Trace}(A)^2 - 4 \det A}}{2} = -2, -5. \]

Eigenvectors can be computed in the usual way by solving the equations

\[ (A - \lambda_j)V_j = 0. \]

We obtain

\[ V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

(b) Setting

\[ T = [V_1 | V_2] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \]

we obtain a matrix that puts \( A \) into its canonical form:

\[ B = T^{-1}AT = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}. \]

(c) The general solution of \( X' = AX \) is

\[ X(t) = c_1 e^{-2t}V_1 + c_2 e^{-5t}V_2 = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{-5t} \\ -c_1 e^{-2t} + c_2 e^{-5t} \end{bmatrix}. \]

The general solution to \( Y' = BY \) is

\[ Y(t) = c_1 e^{-2t}E_1 + c_2 e^{-5t}E_2 = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{-5t} \end{bmatrix}. \]

(d) The phase portrait of \( X' = AX \) is given in Figure 1. The equilibrium is a sink.
3. (25 points) (a) Show that if $X(t)$ is a solution to a linear system $X' = AX$, then so is $Y(t) = X(t + a)$, for any real number $a$.

(b) Suppose that $X_1(t)$ and $X_2(t)$ are both solutions to $X' = AX$. If $X_1(t_1) = X_2(t_2)$, for some real numbers $t_1, t_2$, what is the relation between $X_1$ and $X_2$? In particular, what can be said about the solution curves defined by $X_1$ and $X_2$?

**Solution:** (a) Assume that $X(t)$ is a solution to $X' = AX$. Then $X'(t) = AX(t)$, for all real numbers $t$. Therefore,

$$Y'(t) = \frac{d}{dt}X(t + a)$$
$$= X'(t + a)$$
$$= AX(t + a)$$
$$= AY(t).$$

This proves that $Y(t)$ is also a solution to $X' = AX$.

**Remark.** Observe that the $Y(t)$ and $X(t)$ define the same solution curves; the former is just slightly ahead (if $a > 0$) or behind (if $a < 0$) the latter.

(b) Let $Y_1(t) = X_1(t + t_1)$ and $Y_2(t) = X_2(t + t_2)$. By (a), both $Y_1$ and $Y_2$ are solutions to $X' = AX$. Furthermore,

$$Y_1(0) = X_1(t_1) = X_2(t_2) = Y_2(0).$$

By uniqueness of solutions, $Y_1(t) = Y_2(t)$, for all real numbers $t$. This implies

$$X_1(t + t_1) = X_2(t + t_2),$$
for all \( t \). Or, by a change of variables,
\[
X_1(t + t_1 - t_2) = X_2(t),
\]
for all \( t \). Therefore, \( X_1 \) and \( X_2 \) define the same solution curve.

4. (25 points) A solution \( X(t) \) to a system of differential equations is called \textbf{periodic} if it is not an equilibrium solution (i.e., it is not constant) and there exists a real number \( \tau > 0 \) such that
\[
X(t + \tau) = X(t),
\]
for all \( t \). The smallest such number \( \tau \) is called the \textbf{period} of \( X(t) \).

(a) If
\[
A = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix},
\]
show that all non-equilibrium solutions to \( X' = AX \) are periodic with the same period. Find the period.

(b) If \( Y(t) \) is a periodic solution to \( Y' = BY \), show that \( X(t) = TY(t) \) is a periodic solution to \( X' = AX \), where \( B = T^{-1}AT \) and \( T \) is any matrix with non-zero determinant.

(c) If a matrix \( A \) has purely imaginary eigenvalues, show that every non-equilibrium solution to \( X' = AX \) is periodic with the same period. What is that period?

**Solution:** (a) If \( A \) is in the given form, we know that the general solution to \( X' = AX \) is
\[
X(t) = c_1 \cos \beta t \, E_1 + c_2 \sin \beta t \, E_2,
\]
where \( E_1, E_2 \) are the vectors in the standard basis for \( \mathbb{R}^2 \). The period of \( \cos \beta t, \sin \beta t \) (where, without loss of generality we assume \( \beta > 0 \)) is
\[
\zeta = \frac{2\pi}{\beta}.
\]
Therefore, for every \( c_1, c_2 \), \( X(t + \tau) \equiv X(t) \). This shows that every non-equilibrium solution is periodic with period \( \zeta \).

(b) Suppose that \( Y(t + \tau) = Y(t) \), for all \( t \). Then for all \( t \),
\[
X(t + \tau) = TY(t + \tau) = TY(t) = X(t).
\]
Therefore, \( X(t) \) is periodic with the same period as \( Y(t) \).
(c) Suppose that $A$ has eigenvalues $\pm i\beta$, for some $\beta > 0$. Then there exists an invertible matrix $T$ such that

$$B = T^{-1}AT = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}. $$

It was shown in class that all solutions to $X' = AX$ are of the form $TY(t)$, where $Y(t)$ is a solution to $Y' = BY$. Since, by part (a), every non-equilibrium solution to $Y' = BY$ is periodic with period $2\pi/\beta$, (c) implies that every non-equilibrium solution to $X' = AX$ is periodic with period $2\pi/\beta$. 